Theorem. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ its Cartan subalgebra. Let $x \in \mathfrak{h}$. Then $x$ is semisimple.

Proof: Since $\mathfrak{h}$ is abelian, $(\text{ad}x)(\mathfrak{h}) = \{0\}$. Let $x = s + n$ be the Jordan decomposition of $x$ in $\mathfrak{g}$. Since $\text{ad}x$ is a polynomial in $\text{ad}x$ without constant term $(\text{ad}x)(\mathfrak{h}) = \{0\}$, i.e., $[s, y] = 0$ for any $y \in \mathfrak{h}$. Therefore $\mathfrak{h}' = k \cdot s + \mathfrak{h}$ is an abelian Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ is maximal abelian, $s \in \mathfrak{h}$.
Hence, \( m = x - s \in \mathcal{H} \).

Let \( y = 0 \) if \( x_o \neq 0 \). We proved that \( y \subseteq 0 \) if \( x_o, \lambda \) for any eigenvalue \( \lambda = 0 \) of \( \text{ad} x_o \).

Let \( y \in \mathcal{H} \). Then \( \text{ad} y \) and \( \text{ad} n \) commute. Since \( \text{ad} n \) is nilpotent

\[ \Rightarrow \text{ad} y \text{ad} n \text{ is nilpotent.} \]

Hence, \( B(y, n) = 0 \). It follows that \( B(y, n) = 0 \) for all \( y \in \mathcal{H} \). Since \( B \) is not degenerate, \( n = 0 \) and

\[ x = s. \]
Corollary. Let \( g \) be a semisimple Lie algebra. Let \( x \in g \) be a regular element. Then \( x \) is semisimple.

Proof. \( x \) regular

\[ \Rightarrow y = gJ(x,0) \] is a Cartan subalgebra, \( x \in h \).

\[ \Rightarrow x \] is semisimple. \( \square \)

\[ \text{D}: g \rightarrow g \] is a derivation of \( g \) if

\[ D([x,y]) = [Dx,y] + [x,Dy] \]

for \( x, y \in g \).

\( \text{Der}(g) \) - set of all derivations

\( \text{Der}(g) \subset \mathbb{L}(g) \), \( \text{Der}(g) \) vector
subspace of \( \mathfrak{g}(\mathfrak{g}) \)

\[ D_1, D_2 \in \text{Der}(\mathfrak{g}) \]

\[ [D_1, D_2][[x, y]] = D_1D_2[[x, y]] - D_2D_1[[x, y]] = \]

\[ D_1([D_2x, y] + [x, D_2y]) - D_2([D_1x, y] + [x, D_1y]) = \]

\[ -[D_1D_2x, y] + [D_2x, D_1x] + [D_2y, D_1x] + [x, D_2D_1y] - [D_2D_1x, y] - [D_2y, D_1x] - [x, D_2D_1y] \]

\[ = [[D_1, D_2][x, y]] + [x_1[[D_1, D_2][y]]] \]

\[ [D_1, D_2] \in \text{Der}(\mathfrak{g}) \]

- Derivations are a Lie
A subalgebra of $L(\mathfrak{g})$.
\[ \text{Der}(\mathfrak{g}) \subset L(\mathfrak{g}) \text{ Lie algebra} \]
\[ \text{ad} : \mathfrak{g} \to \text{Der}(\mathfrak{g}) \]
\[ \text{ad} \mathfrak{g} - \text{inner derivations} \]
Inner derivations form an ideal in $\text{Der}(\mathfrak{g})$
\[ x \in \mathfrak{g}, \; D \in \text{Der}(\mathfrak{g}) \]
\[ [D, \text{ad} x](y) = D(\text{ad} x(y)) - \text{ad} x(D(y)) = D([x, y]) - [x, D(y)] \]
\[ = [Dx, y] + [x, Dy] - [x, Dy] = \text{ad}(Dx)(y) \]
\[ [D, \text{ad} x] = \text{ad}(Dx) \]
\[ \implies [D, \text{ad} x] \in \text{ad}(\mathfrak{g}) \]
\[ \implies \text{ad} \mathfrak{g} \subset \text{Der}(\mathfrak{g}) \text{ is an ideal.} \]
Let $\text{Aut}(\mathfrak{g}) \subset \text{GL}(\mathfrak{g})$ be the group of all automorphisms of the Lie algebra $\mathfrak{g}$.

Let $D$ be a derivation of $\mathfrak{g}$.

Assume that $D$ is nilpotent.

\[
e^D = \sum_{p=0}^{\infty} \frac{1}{p!} D^p
\]

is well-defined.

\[
e^D([x,y]) = \sum_{p=0}^{\infty} \frac{1}{p!} D^p([x,y])
\]

By induction

\[
D^p([x,y]) = \sum_{a=0}^{p} \binom{p}{a} [D^a x, D^{p-a} y]
\]

\[
\Rightarrow e^D([x,y]) = \sum_{p=0}^{\infty} \sum_{a=0}^{p} \frac{1}{a!} \frac{1}{p-a} [D^a x, D^{p-a} y]
\]

\[
= [e^D x, e^D y]
\]
\[ \Rightarrow e^D \in \text{Aut}(\mathfrak{g}). \]

\( x \in \mathfrak{g}, \text{ad} x \text{ nilpotent}, e^{\text{ad} x} \in \text{Aut}(\mathfrak{g}) \)

\( \text{Aut}_e(\mathfrak{g}) - \) subgroup of \( \text{Aut}(\mathfrak{g}) \)

generated by \( e^{\text{ad} x}, \text{ad} x \text{ nilpotent.} \)

\( \text{Aut}_e(\mathfrak{g}) - \) elementary automorphisms.

Let \( \varphi \in \text{Aut}(\mathfrak{g}) \)

\[ (\varphi^{-1} \circ \text{ad} x \circ \varphi)(y) = \varphi^{-1}([x, \varphi(y)]) = \]

\[ = [\varphi^{-1}(x), y] = \text{ad} \varphi^{-1}(x) \]

\[ \varphi^{-1} \circ \text{ad} x \circ \varphi = \text{ad} \varphi^{-1}(x) \]

\[ \Rightarrow \varphi^{-1} e^{\text{ad} x} \varphi = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad} x)^n \varphi = \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} (\varphi^{-1} \circ \text{ad} x \circ \varphi)^n = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad} \varphi^{-1}(x))^n = \]

\[ = e^{\text{ad} \varphi^{-1}(x)} \Rightarrow \]
$\text{Aut}_e(cg)$ is a normal subgroup of $\text{Aut}(cg)$. 