Cartan subalgebras in semisimple Lie algebras

Let \( \mathfrak{g} \) be a semisimple Lie algebra over an algebraically closed field \( k \).

Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \). Then there exists \( h_0 \in \mathfrak{h} \) such that

\[
\mathfrak{h} = \mathfrak{g}(h_0, 0).
\]

Moreover, we have

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i=1}^{s} \mathfrak{g}(h_0, \lambda_i)
\]

where \( \lambda_1, \ldots, \lambda_s \) are nonzero
eigenvalues of $h_0$.

For any $y$, we have

$$\text{adh}(\mathfrak{g}(h_0, \lambda_i)) = \mathfrak{g}(h_0, \lambda_i)$$
$$\text{adx}(\mathfrak{g}(h_0, \lambda_i)) = \mathfrak{g}(h_0, \lambda_i + \lambda)$$

$\lambda \in \mathfrak{g}(h_0, \lambda_i)$

$$\text{adx}(\text{adh}(\mathfrak{g}(h_0, \lambda_i))) = \mathfrak{g}(h_0, \lambda_i + \lambda)$$

$$\begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}$$

matrix in a basis

$$\begin{bmatrix}
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus \mathfrak{g}(h_0, \lambda_i)
\end{bmatrix}$$

$B(h_i x) = 0$ for $i = 1, \ldots, s$. Since $h$ is nilpotent $h$ is solvable. $h \equiv \text{ad} y$. By Cartan criterion, it follows that
that $\mathfrak{y} \subseteq \mathfrak{h}$. On the other hand, $\mathfrak{h} \cap \mathfrak{y}$ is orthogonal to $O\mathfrak{g}(h_0, \lambda_i)$. Hence, $\mathfrak{h} \cap \mathfrak{y} = \{0\}$. Since $O\mathfrak{g}$ is semisimple, Killing form is nondegenerate and $\mathfrak{h} \mathfrak{y} = \{0\}$.

Hence, $\mathfrak{y}$ is abelian.

Theorem. Let $O\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field $k$. Let $\mathfrak{y}$ be a Cartan subalgebra of $O\mathfrak{g}$. Then $\mathfrak{y}$ is abelian.

$\Rightarrow \mathfrak{y}$ is maximal abelian.

(since it is maximal abelian since it is maximal nilpotent).
To analyze Cartan subalgebras of semisimple Lie algebras more carefully, we need the following result.

**Theorem.** Let \( \mathfrak{g} \) be a semisimple Lie algebra. Let \( x \in \mathfrak{g} \). Then there exist unique elements \( s, \nu \in \mathfrak{g} \) such that

1. \( \text{ad } s \) is a semisimple linear map, \( \text{ad } \nu \) is a nilpotent linear map
2. \( x = s + \nu \)
3. \( [s, \nu] = 0 \).
$s$ is the semisimple part of $x$ and $n$ is the nilpotent part of $x$.

$x = s + n$ is the Jordan decomposition of $x$ in $G$.

The Jordan decomposition is unique.

$x = s + n \implies \text{ad} x = \text{ad} s + \text{ad} n$

$\Rightarrow \text{ad} s, \text{ad} n$ are unique,

Since $\text{ad}: G \rightarrow \mathfrak{g}(G)$ is injective, $s$ and $n$ are unique!
To prove existence, we need some preparation.

**Lemma.** Let \( \mathfrak{g} \) be a faithful representation of a semisimple Lie algebra \( \mathfrak{g}_f \). Then

\[
\beta(x,y) = tr(g(x)g(y))
\]

is a nondegenerate bilinear form on \( \mathfrak{g} \times \mathfrak{g} \).

(we proved this for \( \mathfrak{g} = \text{ad}! \))

**Proof:**

Let \( s = \{ x \in \mathfrak{g}_f \mid x \perp g(y) \} \). Then \( s \) is an ideal in \( \mathfrak{g}_f \).

\( s \) is orthogonal to itself with respect to \( \beta \). By Cartan criterion
s is solvable, so $s + 3 = 403$.

$\Rightarrow \beta$ is nondegenerate.

Let $\mathfrak{g}$ be a Lie algebra and $B$ its Killing form. Let $\mathfrak{o}_{\mathfrak{g}}$ be a semisimple Lie subalgebra of $\mathfrak{g}$. Let $\mathfrak{h} = \mathfrak{g}^1$ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{o}_{\mathfrak{g}}$ and

$\text{ad}(x)(\mathfrak{h}) < \mathfrak{h}$ for all $x \in \mathfrak{o}_{\mathfrak{g}}$.

Proof: Since $\mathfrak{o}_{\mathfrak{g}}$ is semisimple, $\text{ad} : \mathfrak{o}_{\mathfrak{g}} \to \mathfrak{g}$ is faithful. Hence, $B|_{\mathfrak{o}_{\mathfrak{g}} \times \mathfrak{o}_{\mathfrak{g}}}$ is nondegenerate. It follows that $\mathfrak{h} \cap \mathfrak{o}_{\mathfrak{g}} = \{0\}$.

Moreover, $\dim \mathfrak{h} = \dim \mathfrak{g} - \dim \mathfrak{o}_{\mathfrak{g}}$. 
\[ \therefore q_j = \alpha + \beta. \]

If \( x \in \alpha, y \in \beta \) we have
\[ B([x,y], z) = -B(y, [x,z]) = 0 \]
for all \( z \in \Omega \). Hence, \([x,y] \in \beta \).

Lemma. Let \( \alpha \) be a semisimple ideal in \( q_j \). Then \( \beta \) is an ideal in \( q_j \) and
\[ \beta = \{ x \in q_j | ad_x (\alpha) = 0 \} \leq q_j. \]

In particular \( q_j = q_j \times \Omega \).

Proof. Since \( \alpha \) is an ideal \( \beta \) is also an ideal. Hence
\[ q_j = \beta \times \Omega. \]
Let \( x \in q_j \), \( x = x_1 + x_2 \).
Let $\mathfrak{g}$ be a simple Lie algebra. Let $y \in \mathfrak{g}$.

\[ [x_1, y] = [x_1, y] + [x_2, y] \]

If $[x_1, y] = 0$ for all $y \in \mathfrak{g}$,

\[ [x_1, y] = 0 \implies \text{ad} x_1 |_\alpha = 0 \implies x_1 = 0 \]

$x \in \mathfrak{h}$.

Let $\mathfrak{g}$ be a simple Lie algebra. Let $\mathfrak{g}$.

\[ \text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}) \text{ injective} \]

\[ \mathcal{N} = \{ T \in \mathfrak{gl}(\mathfrak{g}) | (\text{ad} T)(\text{ad} \mathfrak{g}) < \text{ad} \mathfrak{g} \} \]

\[ \mathfrak{g} \cong \text{ad} \mathfrak{g} \]

\[ T \in \mathcal{N}, \ T = S + N \ - \text{Jordan decomposition} \]

\[ \text{ad} T = \text{ad} S + \text{ad} N \]

\[ \text{ad} S \text{ and } \text{ad} N \text{ are polynomials} \]
of \text{ad} T \text{ without constant term}
\Rightarrow (\text{ad} S)(\text{ad} g) \subset \text{ad} g \quad (\text{ad} N)(\text{ad} g) \subset \text{ad} g
\text{ad} g \text{ is an ideal in } N
\begin{aligned}
\text{B Killing form of } N, \quad h = (\text{ad} g)^{-1} \\
n = h \times (\text{ad} g)
\end{aligned}
\text{We have } S, N \in n.

We can write } N = P + R, \text{ P} \in h, \text{ Read } g

Let } \lambda \text{ be an eigenvalue of } P \text{ and } \text{ E its eigenspace. Let } Q \in \text{ad } g.

Then } [P, Q] = 0. \text{ It follows that } Q(E) \subseteq E \text{ since for any } v \in E:

\begin{align*}
PQv &= QPv = \lambda Qv \quad \text{for } v \in E.
\end{align*}
Let $x \in \mathcal{O}_J$ and $Q = \text{ad}x$. Then $\text{ad}x(E) \subseteq E$. Hence $E$ is an ideal in $\mathcal{O}_J$. Since $\mathcal{O}_J$ is simple and $E \neq \{0\}$, we must have $E = \mathcal{O}_J$.

$\Rightarrow P = \lambda I$,

$N = \lambda I + R$

$R = \text{ad}z, z \in \mathcal{O}_J$. Since $\mathcal{O}_J = \mathcal{O}_J$, $z \in \mathcal{O}_J$

$\Rightarrow H \text{ad}z = 0$

$0 + fN = \lambda \cdot \dim \mathcal{O}_J + 0 = \lambda \cdot \dim \mathcal{O}_J$

$\Rightarrow \chi = 0, \quad N = R \in \text{ad} \mathcal{O}_J$.

Assume that $T = \text{ad}x$. Then $\text{ad}x = S + N = S + \text{ad}z, z \in \mathcal{O}_J$.
\[ W = \text{ad} z \quad S = \text{ad} (x - z) \in \text{ad} \mathfrak{g} \]

\[ x = s + n, \quad \text{ad} x = S, \quad \text{ad} n = N \]

\[ \text{ad} ([s, n]) = [\text{ad} s, \text{ad} n] = [s, N] = 0 \]

\[ \Rightarrow [s, n] = 0. \]

This proves the existence for simple \( \mathfrak{g} \).

Assume that \( \mathfrak{g} \) is semisimple.

\[ \mathfrak{g} = M_1 \times \cdots \times M_k \]

\( M_i \) - minimal ideals - simple.

\[ x \in \mathfrak{g}, \quad x = x_1 + \cdots + x_k, \quad x_i \in M_i \]

\[ x_i = s_i + n_i, \quad [s_i, n_i] = 0 \quad \text{ad} s_i \quad \text{semisimple, ad} n_i \quad \text{nilpotent} \]

\[ s_i, n_i \in M_i \]
\[ [s_i, s_j] = 0 \Rightarrow s = s_1 + \ldots + s_k \]

is semisimple

\[ [n_i, n_j] = 0 \Rightarrow n = n_1 + \ldots + n_k \]

is nilpotent

\[ [s_i, n_i] = 0, \quad [s_i, n_j] = 0 \quad i \neq j \]

\[ \Rightarrow [s, n] = 0 \]

\[ x = (s_1 + \ldots + s_k) + (n_1 + \ldots + n_k) = s + n \]

- Jordan decomposition of \( x \).