\( h_0 \in \mathfrak{g} \) - regular element
\[ h = \mathfrak{g}(h_0, 0) \] Lie algebra
we proved that \( h \) is nilpotent
Lie subalgebra of \( \mathfrak{g} \).

Lemma. The Lie algebra \( h \)
is equal to its own normalizer.

Proof.

\[ n = \{ x \in \mathfrak{g} \mid (\text{ad}x)(h) \subset h \} \]

This is a Lie subalgebra, \( h \subset n \).
\[ x \in n, \quad (\text{ad}x)(h_0) \subset h \implies (\text{ad}h_0)(x) \subset h, \quad (\text{ad}h_0)(n) \subset h, \quad (\text{ad}h_0)^n(h) = \{0\} \text{ since } h \text{ is nilpotent} \]

\[ \implies (\text{ad}h_0)^{n+1}(n) = \{0\} \]

\[ \implies n = h. \]
Def. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a Cartan subalgebra if

(a) $\mathfrak{h}$ is nilpotent,
(b) $\mathfrak{h}$ is equal to its own normalizer.

We proved that any Lie algebra over $k$ contains a Cartan subalgebra.

Theorem. Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$. Then $\mathfrak{h}$ is a maximal nilpotent Lie subalgebra in $\mathfrak{g}$.

Proof. Let $\mathfrak{h}$ be a nilpotent Lie subalgebra of $\mathfrak{g}$. $\text{ad}_x$ are nilpotent for $x \in \mathfrak{h}$,

$\Rightarrow \text{ad}_x$ are nilpotent for $x \in \mathfrak{h}$,
Assume that $\mathfrak{n} \neq \mathfrak{h}$.

Then $\mathfrak{h}$ acts on $\mathfrak{n}/\mathfrak{h}$ by nilpotent linear transformation. Fix $\mathfrak{n}/\mathfrak{h}$ such that $h \cdot v = 0$ for $h \in \mathfrak{h}$.

$\Rightarrow \mathfrak{n} = x + \mathfrak{h}$, $x \in \mathfrak{n}$, $x \notin \mathfrak{h}$,

$[h, x] \in \mathfrak{h}$

$\Rightarrow x$ is in the normalizer of $\mathfrak{h}$.

Since $\mathfrak{h}$ is a Cartan subalgebra $x \notin \mathfrak{h}$ $\Rightarrow \mathfrak{n} = 0$. Contradiction!

$\Rightarrow \mathfrak{n} = \mathfrak{h}$. \[\square\]

Maximal nilpotent Lie subalgebras do not have to be Cartan subalgebras.
Example: $g = \mathfrak{sl}(2, \mathbb{C})$

$$n = \left\{ \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \right\} \quad \text{abelian} \Rightarrow \text{nilpotent}$$

if it is maximal nilpotent, since two dimensional Lie algebras are either abelian or solvable. Normalizer is $\left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}$, it is solvable and not abelian.

Theorem. Let $g_j$ be a Lie algebra over an algebraically closed field $k$. Let $h_j$ be a Cartan subalgebra of $g_j$. Then there exist $h \in h_j$ such that $h = g_j(h, 0)$.
Proof: The adjoint representation of $g$ restricted to $h$ defines a representation of $h$ on $g$. Clearly, $h$ is invariant subspace for this representation $\Rightarrow$ We get a representation of $h$ on $G/\gamma$. Denote it by $\sigma$. Assume that there exists $h \in h$ such that $\sigma(h)$ is invertible. Then the nilspace of $ad_h$ has to be a subspace of $h$, i.e., $g(h,0) \subset h$. Since $h$ is nilpotent, $ad_h(h)$ is nilpotent.
Hence, $\gamma < \mathfrak{g}(h, \sigma)$. It follows that $\gamma = \mathfrak{g}(h, \sigma)$. Since $\gamma$ is nilpotent, it is solvable. Hence, by Lie's theorem, there is a flag 

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathfrak{g}/\gamma$$

dim $V_i = i$; of $\gamma$-invariant subspaces. We have

$$\sigma(x)\sigma - \alpha_i(x)\sigma \in V_{i-1}$$

for $x \in V_i$, $x \in \gamma$; where

$$\alpha_i(\cdots) \in \mathfrak{g}^*.$$

$$\det \sigma(x) = \prod_{i=1}^n \alpha_i(x)$$

Hence, it is enough to find $\mathfrak{h}$.
such that \( a_i(h) = 0 \) for all \( 1 \leq i \leq m \).

Assume that such \( h \) does not exist. Then \( a_1, a_2, \ldots, a_m \) is a polynomial \( P \) on \( h \) such that \( P(h) = 0 \) for all \( h \).

Lemma. Let \( P \in k[x_1, \ldots, x_m] \) such that \( P(x_1, \ldots, x_m) = 0 \) for all \( (x_1, \ldots, x_m) \in k^m \). Then \( P = 0 \).

We postpone the proof of the lemma.

Hence, \( a_1 \cdot a_2 \cdot \ldots \cdot a_m = 0 \). This implies that there exists \( 1 \leq i \leq m \)
such that $\alpha_i = 0$.

Assume that $\alpha_1 \neq 0, \alpha_2 \neq 0, \ldots, \alpha_{k-1} \neq 0$ and $\alpha_k = 0$. Then there exists $x \in y$ such that $\alpha_1(x) \neq 0, \alpha_2(x) \neq 0, \ldots, \alpha_{k-1}(x) \neq 0$ and $\alpha_k(x) = 0$.

Hence, $\sigma(x)|_{V_k} : V_{k-1} \rightarrow V_{k-1}$ is an isomorphism. Let $V' \subset V_k$ be the kernel of $\sigma(x)|_{V_k}$. Then $V' \neq \{0\}$. Hence, $V_k = V_{k-1} \oplus V'$.

Let $v' \in V'$, $v' \neq 0$.

Claim. $\sigma(y)v' = 0$ for all $y \in y$.

Proof. We first prove

$\sigma(x)^t \sigma(y)v' = \sigma((\text{ad} x)^t y)v'$. 
\[ p = 1 \]

\[ \sigma(x) \sigma(y) \nu' = \sigma(x) \sigma(y) \nu' \]
\[ - \sigma(y) \sigma(x) \nu' = [\sigma(x), \sigma(y)] \nu' = \]
\[ = \sigma([x,y]) \nu' = \sigma((ax)(y)) \nu'. \]

\( p > 1 \) induction step

\[ \sigma(x)^p \sigma(y) \nu' = \sigma(x)^p \sigma(x) \sigma(y) \nu' = \]
\[ = \sigma(x)^p \sigma((ax)(y)) \nu' = \sigma(x)^p \sigma([x,y]) \nu' \]
\[ = \sigma((ax)^p [x,y]) \nu' = \sigma((ax)^{p+1} y) \nu'. \]

This proves the claim.

Since \( b \) is nilpotent,

\[ (ad)(ax)^p y = 0 \text{ for large } p \text{ and all } y \in b. \]

Hence, \( \sigma(x)^p \sigma(y) \nu' = 0 \)

\( \sigma(y) \nu' \) is in the nilspace of
\(a(x)|V_k\). This must be \(V'\).

Therefore, \(\sigma(y)v' \in V'\) for all \(y \in V\).

On the other hand,

since \(\alpha_k = 0\), \(\sigma(y)(V_k) \subset V_{k-1}\),

\[\Rightarrow \sigma(y)v' \subset V_{k-1} \Rightarrow \sigma(y)v' = 0\]

for all \(y \in V'\).

\(v' = z + y\), \(z \in \sigma_y\)

\(0 = \sigma(y)v' = [y, z] + y\)

\(\Rightarrow \exists z \neq y \text{ such that } [z, y] \in \mathcal{H}\)

for all \(y \in \mathcal{V}\) . \((\text{ad} z)(y) > y\).

\(z\) is in the normalizer of \(y\).

Since \(y\) is self-normalizing

\(\Rightarrow\) contradiction! \(\Box\)
Proof of the lemma.

Induction in $n$. \[
\text{\quad} m = 1
\]

$k$ is infinite. No zero polynomial has finitely many zeros.

\[
P(x_1, \ldots, x_n) = \sum_P Q_p (x_1, \ldots, x_{n-1}) X_p^m
\]

Fix $(x_1, \ldots, x_{n-1})$. By 1-dim. result

$Q_p (x_1, \ldots, x_{n-1}) = 0$ for all $(x_1, \ldots, x_{n-1})$.

$\Rightarrow Q_p = 0$.

**Corollary.** dim $\mathfrak{h} \geq \text{rank } \mathfrak{g}$ for any Cartan subalgebra of $\mathfrak{g}$. 