Let \( g \) be a semisimple Lie algebra. Let \( \mathfrak{o} \) be an ideal in \( g \). We proved that \( \mathfrak{o} \) is semisimple. Then \( \mathfrak{o}^+ \) is also an semisimple ideal and
\[
\mathfrak{o}^g = \mathfrak{o} \oplus \mathfrak{o}^+.
\]
Assume that \( \mathfrak{o} \) is a minimal ideal. It is not abelian. Let \( \mathfrak{b} \) be an ideal in \( \mathfrak{o} \). Since \( \mathfrak{o}^g = \mathfrak{o} \oplus \mathfrak{o}^+ \), \( \mathfrak{b} \) is an ideal in \( \mathfrak{o}^g \). \( \Rightarrow \mathfrak{b} = \{0\}, \mathfrak{o} \). \( \mathfrak{o} \) is a simple ideal.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in $\mathfrak{g}$ and $\mathfrak{a} \cap \mathfrak{b} = \{0\}$.

$x \in \mathfrak{a}, y \in \mathfrak{b} \implies (\text{ad}_x \circ \text{ad}_y)(\mathfrak{g}) \subseteq \mathfrak{a} \cap \mathfrak{b} = \{0\}$.

$\text{ad}_x \circ \text{ad}_y = 0 \implies B(x,y) = 0$.

$\mathfrak{a} \perp \mathfrak{b} \implies \mathfrak{b} \subseteq \mathfrak{a}^{\perp}$.

Let $m_1, \ldots, m_p$ be minimal ideals in $\mathfrak{g}$. Then $M_i$ are simple. If $M_i \neq M_j$, $M_i \perp M_j$.

Put $\mathfrak{a} = m_1 \oplus m_2 \oplus \cdots \oplus m_p$.

Firstly, $p \leq \dim \mathfrak{g}$. Hence, $\mathfrak{g}$ has finitely many minimal ideals. Hence, we can assume
that \( M_1, \ldots, M_p \) are all minimal ideals. Then \( \mathfrak{g}_f = \mathfrak{o}_e \oplus \mathfrak{o}_f \).

If \( \mathfrak{o}_f \neq \{0\} \), \( \mathfrak{o}_f \) contains a minimal ideal - one of \( M_i \); and we have a contradiction.

So, \( \mathfrak{o}_f = \{0\} \) and \( \mathfrak{g}_f = \mathfrak{o}_f = M_1 \oplus \ldots \oplus M_p \).

**Theorem.** A semisimple Lie algebra is a product of simple Lie algebras.
Cartan subalgebras

Let Lie algebra over algebraically closed field \( k \). Pick \( h \in \mathfrak{g} \)

\[ \text{ad } h : \mathfrak{g} \to \mathfrak{g}, \mu \in k \]

\[ \mathfrak{g}(h, \mu) = \{ x \in \mathfrak{g} \mid (\text{ad } h - \mu)x = 0 \} \]

for \( p > \| \mathfrak{g} \| \)

\[ \mathfrak{g} = \bigoplus_{i=0}^n \mathfrak{g}(h, \mu_i) \]

\( \mathfrak{m}_0 = 0, \mu_1, \ldots, \mu_n \)

\( \mathfrak{g}(h, 0) \in \mathfrak{h}, \text{ so } \mathfrak{g}(h, 0) \neq \{0\} \).

Lemma: \( x \in \mathfrak{g}(h, \lambda), y \in \mathfrak{g}(h, \mu) \)

\[ \Rightarrow [x, y] \in \mathfrak{g}(h, \lambda + \mu), \]

Proof: \( (\text{ad } h - (\lambda + \mu))[x, y] = \)
\[
\begin{align*}
\text{By induction in } m \\
\left( \text{ad } h - (\lambda + \mu) \right)^m [x, y] &= \\
= \sum_{j=0}^{m} \binom{m}{j} \left[ (\text{ad } h - \lambda)^j x, (\text{ad } h - \mu)^{m-j} y \right] \\
\Rightarrow [g(h, \lambda), g(h, \mu)] &= \mathcal{O}_j(h, \lambda + \mu) \\
\Rightarrow (i) \quad g(h, 0) \text{ is a Lie subalgebra of } \mathcal{O}_j.
\end{align*}
\]

\[ P(h, \lambda) = \det(\lambda I - \text{ad } h) = \]
\[ = \sum_{p=0}^{n} a_p(h) \lambda^p, \quad n = \dim \mathfrak{g} \]

\( a_p \) are polynomials on \( \mathfrak{g} \) with values in \( k \).

\[ a_m = 1, \quad a_p = 0, \quad p = 0, 1, \ldots, n - 1 \]

\[ a_0 \neq 0. \]

Definition: \( r = \text{rank} \mathfrak{g} \)

\[ P(h, \lambda) = \lambda^n + \ldots + a_n(h) \lambda^n \]

\[ \dim \mathfrak{g}(h, 0) \geq r = \text{rank} \mathfrak{g} \]

\[ 0 \leq \text{rank} \mathfrak{g} \leq \dim \mathfrak{g} \]

\[ \text{rank} \mathfrak{g} = \dim \mathfrak{g} \implies P(h, \lambda) = \lambda^n \]

\[ \implies \text{ad } h \text{ is nilpotent for any } h \in \mathfrak{g} \]

\[ \implies \mathfrak{g} \text{ is nilpotent!} \]
If \( \mathfrak{g} \) is not nilpotent

\[ \text{rank of } \mathfrak{g} < \dim \mathfrak{g}. \]

**Def.** \( h \in \mathfrak{g} \) is regular if \( \mathfrak{r}(h) \neq 0 \).

Regular elements form a dense Zariski open set in \( \mathfrak{g} \).

**Ant(\( \mathfrak{g} \))** - group of automorphisms of \( \mathfrak{g} \)

\[ \varphi \in \text{Ant}(\mathfrak{g}), \quad (\text{ad} \varphi(h))(\varphi(x)) = \]

\[ = [\varphi(h), \varphi(x)] = \varphi([h,x]) = \]

\[ = (\varphi \circ \text{ad} h \circ \varphi^{-1})(\varphi(x)) \]

\[ \text{ad} \varphi(h) = \varphi \circ \text{ad} h \circ \varphi^{-1} \]

\[ P(h, \lambda) = P(\varphi(h), \lambda). \]

\( h \) regular \( \iff \) \( \varphi(h) \) is regular
The set of reg of all regular elements is $\text{Aut}(\mathfrak{g})$-invariant.

Fix regular element $h_0 \in \mathfrak{g}$.

$$h = \mathfrak{g}(h_0, 0)$$

is a Lie subalgebra of $\mathfrak{g}$, $\dim h = 1$.

**Lemma**: $h$ is a nilpotent Lie algebra.

**Proof**: $\mathfrak{g} = \bigoplus_{i=0}^{\infty} \mathfrak{g}(h_0, \lambda^i) = h \bigoplus_{i=1}^{\infty} \mathfrak{g}(h_0, \lambda^i)$$

$$= \mathfrak{g}$$

$$[h, \mathfrak{g}(h_0, \lambda^i)] \subset \mathfrak{g}(h_0, \lambda^i) \Rightarrow$$

$$[h, \mathfrak{g}] \subset \mathfrak{g}$$
\( \sigma_1 \) is invariant for \( \text{ad} \, h \).

\( p \) restriction of \( \text{ad} \, h \) to \( \sigma_1 \).

\( g(h_0) \) has nonzero eigenvalues on \( \sigma_1 \).

\( \Rightarrow \ h \mapsto \det g(h) \) is a polynomial function on \( Y \) which doesn't vanish at \( h_0 \), \( \Rightarrow \) nonzero.

Let \( h \in Y \) such that \( \det g(h) \neq 0 \).

Then all eigenvalues of \( \text{ad} \, h \) on \( \sigma_1 \) are nonzero \( \Rightarrow \sigma_1(h,0) \subset Y \).

Since \( \dim \sigma_1(h,0) \geq r = \dim Y \)

\( \Rightarrow \sigma_1(h,0) = Y \). \( \text{ad} \, h|_Y \) is nilpotent.

\( (\text{ad}_Y h)^t = 0 \) - matrix coefficients of \( \text{ad}_Y h \) are linear functions on \( Y \).

\( \Rightarrow \) matrix coefficients of \( (\text{ad}_Y h)^t \)
are polynomials on $y$ vanishing on the Zariski open set $\{ \det p(h) \neq 0 \}$. Hence they are $O$ on $y$.

$\Rightarrow \text{ad}_y h'$ are nilpotent for all $h' \in \mathfrak{h} \Rightarrow h$ is nilpotent.