**Cartan's criterion for solvability**

**Theorem (Cartan)** Let $V$ be a vector space over an algebraically closed field $k$. Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Define

$$\beta: \mathfrak{g} \times \mathfrak{g} \to k$$

by $\beta(T_1, S) = \text{tr}(TS)$. Then the following conditions are equivalent:

(i) $\mathfrak{g}$ is solvable;

(ii) $\mathfrak{g}$ is solvable with respect to $\beta$.

**Proof:**

(i) $\Rightarrow$ (ii) If $\mathfrak{g}$ is solvable by Lie's theorem, there is a basis
Let $T \in \mathcal{D}(V)$.

Then $T$ is upper triangular with zeros on diagonal.

Hence, for $S \in \mathcal{D}(V)$, $T \in \mathcal{D}(V)$

$S \cdot T$ is upper triangular with zeros on diagonal. Hence

$tr(S \cdot T) = 0 \Rightarrow \beta(S, T) = 0.

\mathcal{D}(V) = \mathcal{D}(V).

(ii) $\Rightarrow$ (i) \[ [g, g] \cdot g \subseteq \mathcal{L}(V) \],

$\mathcal{G} = \{ T \in \mathcal{L}(V) \mid (\text{ad}T)(g) \subseteq [g, g] \}.$

Let $T \in \mathcal{G}$. Then $[T, A] \in \mathcal{D}(V)$
for any $A \in \mathfrak{g}$.

Moreover, if $A, B \in \mathfrak{g}$, we have
\[
\text{tr}([A, B] \cdot T) = \text{tr}(ABT) - \text{tr}(BAT)
\]
\[
= \text{tr}(ABT) - \text{tr}(ATB) = \text{tr}(A [B, T]).
\]
If $T \in \mathfrak{g}$, $[B, T] = -\text{ad}_T(B) \in \mathfrak{g}$

for any $B \in \mathfrak{g}$.

If $g \perp \mathfrak{g}$ \implies $\text{tr}(A [B, T]) = 0$

\[
\implies \text{tr}([A, B] \cdot T) = 0
\]

By the lemma, this implies that $[A, B]$ is nilpotent. Therefore, any element of $\mathfrak{g}$ is nilpotent.

Hence $\mathfrak{g}$ is a nilpotent Lie algebra.
Therefore we have

\[ 0 \to \mathfrak{d} \to \mathfrak{o} \to \mathfrak{o}/\mathfrak{d} \to 0 \]

\[ \mathfrak{m}' \text{ nilpotent} \quad \mathfrak{a} \text{ abelian} \]

\[ \text{solvable} \]

Hence, \( \mathfrak{o}/\mathfrak{a} \) is a solvable Lie algebra.

Theorem. Let \( \mathfrak{o} \) be a Lie algebra and \( \mathfrak{a} \) its radical.

Then

\[ \mathfrak{r} = (\mathfrak{d} \mathfrak{o})^+ \]

with respect to the Killing form.
Proof. Assume first that \( \pi \) is an irreducible representation of \( \mathfrak{g} \). Then \( \pi |_{\mathfrak{g}} = \chi(\cdot)I \).

Hence, for \( x \in \mathfrak{g}, y, z \in \mathfrak{g} \):

\[
\tau (\pi(x), \pi([y, z])) = \\
= \chi(x) \tau (\pi(y), \pi(z)) = 0 \\
= \beta(x, y) \tau (\pi(x), \pi(y)) = 0
\]

for \( x \in \mathfrak{g} \) and \( y \in \mathfrak{so} \).

**Induction in limit**:

\[
\tau (\pi(x), \pi(y)) = 0, \quad x \in \mathfrak{r}, \quad y \in \mathfrak{so}
\]

for any finite-dimensional representation \( (\pi, V) \).
If \( \dim \pi \) is minimal, \( \pi \) is irreducible. Otherwise, \( V \) has a minimal invariant subspace \( U \)

\[
0 \rightarrow U \rightarrow V \rightarrow V/U \rightarrow 0
\]

\[
\pi_U \quad \pi \quad \pi_{V/U}
\]

irreducible

\[
\dim (V/U) < \dim V.
\]

\[
\pi_U \quad \pi_U \quad \pi_U
\]

\[
0 = \text{tr} (\pi(x) \pi(y)) = \text{tr}(\pi_U(x) \pi_U(y)) + \text{tr}(\pi_{V/U}(x) \pi_{V/U}(y))
\]

= 0 by first step & induction assumption.

This proves our claim.
Applying this statement to the adjoint representation, we get the statement for the Killing form,

\[ B(x, y) = 0 \text{ for } x \in \mathfrak{r}, y \in \mathfrak{do}. \]

Hence, \( r \subset (\mathfrak{do})^\perp \).

\( r' \) is an ideal.

For \( x \in r', y \in \mathfrak{o}, z \in \mathfrak{do} \)

\[ B([y, x], z) = -B(x, [y, z]) = 0 \]

\[ \Rightarrow [x, y] \in (\mathfrak{do})^\perp \Rightarrow [x, y] \in r'. \]

\( r' \) is an ideal.
8

\( \mathfrak{g} \) - Lie algebra, \( \mathfrak{h} \) - subalgebra, ideal

\[
\text{ad}_x = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} \quad \text{for } x \in \mathfrak{h}
\]

\[
B_{\mathfrak{g}}(x, y) = (\text{ad}_x \text{ad}_y) = \tau_x (\text{ad}_y x \cdot \text{ad}_y y) = B_{\mathfrak{h}}(x, y), \quad x, y \in \mathfrak{h}.
\]

2. \( \mathfrak{h}' \) is solvable.

By the above argument

\[
B_{\mathfrak{g}}(x, y) = B_{\mathfrak{h}'}(x, y), \quad x, y \in \mathfrak{h}'.
\]

\[
\Rightarrow \quad B_{\mathfrak{h}'} = B_{\mathfrak{g}} |_{\mathfrak{h}' \times \mathfrak{h}'}.
\]

This implies that

\[ \mathfrak{h}' \perp \mathfrak{d}_{\mathfrak{h}'} \]

By Cartan's criterion, \( \text{ad}_{\mathfrak{h}'} \) is
a solvable Lie algebra.

\[ 0 \to z \to r' \to \text{ad}^+ r' \to 0 \]

\[ \text{center of } r' \]

\[ z \text{ is abelian Lie algebra} \]

\[ \text{ad}^+ r' \text{ is solvable Lie algebra} \]

\[ \implies r' \text{ is solvable Lie algebra}. \]

Therefore \( r' \) is a solvable ideal in \( o' \). Hence, \( r' \leq r \).

This implies that \( r' = r \).
**Semisimple Lie algebras**

$k$ - algebraically closed field

**Theorem**: Let $\mathfrak{g}$ be a Lie algebra over $k$. Then the following conditions are equivalent:

(i) $\mathfrak{g}$ is semisimple;

(ii) $B$ is nondegenerate on $\mathfrak{g} \times \mathfrak{g}$.

If these conditions hold, $D\mathfrak{g} = \mathfrak{g}$.

**Proof**: (i) $\Rightarrow$ (ii) $\mathfrak{g}$ is semisimple

$\Rightarrow$ radical $+ \mathfrak{g}$ of $\mathfrak{g}$ is $\{0\} \Rightarrow (D\mathfrak{g})^+ = \{0\}$.

$\mathfrak{g}^+ = \{0\} \Rightarrow$ this means that $B$ is nondegenerate. Hence

$D\mathfrak{g} = ((D\mathfrak{g})^+)^+ = (\{0\}^+)^+ = \mathfrak{g}$. 

(ii) ⇒ (i) Assume that Killing form is nondegenerate. Let $\mathfrak{g}$ be an abelian ideal in $\mathfrak{g}$.

$x \in \mathfrak{g}, y \in \mathfrak{g}$

$$\text{ad} x \text{ad} y (\mathfrak{g}) = \text{ad} x (\mathfrak{g}) \subseteq \mathfrak{g}$$

$$\text{ad} x \text{ad} y (\mathfrak{g}) = \{0\}$$

$$(\text{ad} x \text{ad} y)^2(\mathfrak{g}) = \{0\}$$

$$(\text{ad} x \text{ad} y)^2 = 0 \Rightarrow \text{ad} x \text{ad} y \text{ is nilpotent}, \quad \text{to } (\text{ad} x \text{ad} y)^2 = 0.$$  

$\mathfrak{g} \perp \mathfrak{g}$. Since $\mathfrak{B}$ is nondegenerate, $\mathfrak{g} = \{0\}$. Hence, $\mathfrak{g}$ is semisimple.
Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{o}$ an ideal in $\mathfrak{g}$. Then $\mathfrak{o}^\perp$ is an ideal in $\mathfrak{g}$. $\mathfrak{b} = \mathfrak{o} \cap \mathfrak{o}^\perp$ is an ideal in $\mathfrak{g}$. $\mathfrak{b}$ is solvable with respect to Killing form. By Cartan criterion, $\text{ad} \mathfrak{b}$ is solvable. Since the center of $\mathfrak{g}$ is $\{0\}$, $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{g}(\mathfrak{g})$ is injective. $\Rightarrow \mathfrak{b}$ is solvable $\Rightarrow \mathfrak{b} = \{0\}$. Hence, $\mathfrak{o} \cap \mathfrak{o}^\perp = \{0\}$. $\mathfrak{g} = \mathfrak{o} \oplus \mathfrak{o}^\perp$. $\mathfrak{B}_\alpha = \mathfrak{B}_\mathfrak{g}|_{\alpha \times \mathfrak{g}}$
Let $x \in \mathfrak{a}$, $B_\mathfrak{a}(x, y) = 0$ for $y \in \mathfrak{a}$.

$\Rightarrow B_{\mathfrak{a}^+}(x, y) = 0 \Rightarrow B_{\mathfrak{a}^+}(x, y) = 0$ for all $y \in \mathfrak{a}^+$.

$\Rightarrow B_\mathfrak{a}(x, y) = 0$ for all $y \in \mathfrak{a}$.

$\Rightarrow x = 0$.

$B_\mathfrak{a}$ is nondegenerate. Hence $\mathfrak{a}$ is semisimple.

In addition, $\mathfrak{a}^+$ is semisimple.

$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^+$. 