Jordan decomposition

Let $V$ be a vector space over $k/\text{alg. closed}

$T \in \mathcal{L}(V)$ is semisimple if there exists a basis $e_1, \ldots, e_n$ of $V$ such that $Te_i = \lambda_i e_i$, $\lambda_i \in k$.

$T \in \mathcal{L}(V)$, $T = S + N$

$S$ semisimple

$N$ nilpotent

$[S, N] = 0$.

Extra. $S$ and $N$ are given by polynomials in $T$ without constant term.
Uniqueness:

\[ T = S + N = S' + N' \]

\[ S = P(T) \quad N = Q(T) \]

\[ [S, N] = 0 \quad [T, S] = [T, N] = 0 \]

\[ S + N = S' + N' \]

\[ S - S' = N' - N \]

\[ S' \text{ commutes with } S' \text{ and } N' \Rightarrow \]

\[ \text{with } T \Rightarrow \text{ with } S \]

\[ \Rightarrow S - S' \text{ is semisimple} \]

Analogously, \( N' \) commutes with \( T \Rightarrow \text{ with } N \Rightarrow N - N' \text{ is nilpotent}. \]

\[ \Rightarrow S - S' = N' - N \]

is semisimple & nilpotent \( \Rightarrow \) equal to 0.
Jordan decomposition is unique!

- $S$ is the **semisimple part**
- $N$ is the **nilpotent part** of $I$

$V$ vector space over $k$
$e_1, \ldots, e_m$ basis of $V$
$E_{ij} \in \mathcal{L}(V)$, $E_{ij}e_k = 0$ $k \neq j$

$E_{ij}e_j = e_i$

$A e_j = \sum_{i=1}^n A_{ij} e_i$

$A e^j = \sum_{i=1}^n A_{ij} E_{ij} e_j = \left( \sum_{i=1}^n \sum_{k=1}^n A_{ik} E_{ik} \right) e_j$

$(E_{ij} ; 1 \leq i, j \leq m)$ is a basis of $\mathcal{L}(V)$. 
\[ T \in \mathfrak{gl}(V) \] 
\[ T e_i = \lambda_i e_i \]
\[ (\text{ad}T)(E_{ij}) = [T, E_{ij}] = \]
\[ = TE_{ij} - E_{ij}T \]
\[ ((\text{ad}T)(E_{ij})) e_k = TE_{ij}e_k - \lambda_i E_{ij} e_k = 0 \]
\[ = 0 \]

if \( k \neq j \), if \( k = j \)
\[ ((\text{ad}T)(E_{ij})) e_j = T e_i - \lambda_j E_{ij} e_j = \]
\[ = (\lambda_i - \lambda_j)e_i = (\lambda_i - \lambda_j)E_{ij} e_j \]
\[ \Rightarrow (\text{ad}T)(E_{ij}) = (\lambda_i - \lambda_j)E_{ij} \]
\[ \Rightarrow \text{ad}T \text{ is semisimple} \]

We proved that \( T \) nilpotent \( \Rightarrow \)
\( \text{ad}T \) is nilpotent.

\[ T = S + N \Rightarrow \text{ad}T = \text{ad}S + \text{ad}N \]

Jordan decomposition

Jordan decomposition
Lemma. Let $V$ be a vector space over algebraically closed field $k$. Let $U \subset W$ be two subspaces of $L(V)$.

Put $Y = \{ T \in L(V) \mid (\text{ad} T)(W) \subset U \}$. Let $A \in Y$ be such that $(\text{ad} AB) = 0$ for all $B \in Y$. Then $A$ is nilpotent.

Proof. $B \in Y$. Then

$(\text{ad} B)(W) \subset U \subset W$ and

$(\text{ad} B)(U) \subset U$

$\Rightarrow V$ and $W$ are invariant for $\text{ad} B$.

Assume that $A \in Y$, $tr(AB) = 0$ for all $B \in Y$. 
Let \( A = S + N \) - Jordan decomposition.

\( e_1, \ldots, e_n \) basis of \( V \) \( Se_i = \lambda_i e_i \).

Let \( L \) be a vector subspace of \( V \) over \( \mathbb{Q} \) spanned by \( \lambda_1, \ldots, \lambda_m \).

Let \( f : L \to \mathbb{Q} \) be a \( \mathbb{Q} \)-linear form on \( L \).

Put \( T e_i = f(\lambda_i) e_i, \ 1 \leq i \leq n \).

\[
(adT)(E_{ij}) = (f(\lambda_i) - f(\lambda_j)) E_{ij}
\]

\( \lambda_i - \lambda_j \in L \quad \lambda_i - \lambda_j = \lambda_p - \lambda_g \)

\[
f(\lambda_i) - f(\lambda_j) = f(\lambda_i - \lambda_j) =
\]

\[
f(\lambda_p - \lambda_g) = f(\lambda_p) - f(\lambda_g)
\]

\( \lambda_i - \lambda_j = 0 \quad f(\lambda_i) - f(\lambda_j) = 0 \)
There exists $P \in k[x]$ such that
\[ P(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j) \]
and $P$ has no constant term.

It follows that
\[ P(\text{ad} S) E_{ij} = P(\lambda_i - \lambda_j) E_{ij} = (f(\lambda_i) - f(\lambda_j)) E_{ij} = (\text{ad} T) E_{ij} \]
\[ \Rightarrow P(\text{ad} S) = \text{ad} T \]

On the other hand, $\text{ad} S = Q(\text{ad} A)$ with no constant term.

\[ \text{ad} T = (P \circ Q)(\text{ad} A) \]
\[ \Rightarrow T \in \mathfrak{g}! \]

This implies that $\text{tr}(\text{ad} T) = 0$. 
Let $V_i$ be the eigenspace of $S$ for eigenvalue $\lambda_i$. Then it is invariant for $N$. Moreover, $T$ act by multiplication by $f(\lambda_i)$ on $V_i$. Therefore, $V_i$ is invariant for $T$ and $A$ and

$$tr(AT|_{V_i}) = tr(A|_{V_i} \cdot T|_{V_i}) =$$

$$tr(S|_{V_i} \cdot T|_{V_i}) + tr(N|_{V_i} \cdot T|_{V_i}) =$$

$$= tr(S \cdot T|_{V_i})$$

$$= 0$$

$$\Rightarrow \ tr(AT) = tr(ST) =$$

$$\sum_{i=1}^{n} \lambda_i f(\lambda_i) \Rightarrow \sum_{i=1}^{n} \lambda_i f(\lambda_i) = 0.$$

Now $b f\left(\sum_{i=1}^{n} \lambda_i f(\lambda_i)\right) = \sum_{i=1}^{n} f(\lambda_i)^2$.
Since \( f(\lambda_i)^2 \) are positive, this implies that \( f(\lambda_i) = 0 \). Hence \( f \) must be \( 0 \). Since \( f \) was arbitrary, \( L \) must be \([0]^J\).

Hence, all eigenvalues \( \lambda_1, \ldots, \lambda_m \) are \( 0 \). Hence, \( S = 0 \) and \( A = N \) is nilpotent. \( \Diamond \)