Lie's theorem

Let \( \mathfrak{g} \) be a Lie algebra over an algebraically closed field \( \mathbb{K} \).
Let \( V \) be a finite-dimensional vector space over \( \mathbb{K} \).

A Lie algebra morphism
\[
\pi : \mathfrak{g} \rightarrow \mathfrak{L}(V)
\]

is called a representation of \( \mathfrak{g} \) on \( V \).

The representation \( \pi \) is irreducible, if there is no nontrivial vector space \( U \subset V \) such that \( \pi(x)(U) \subset U \) for all \( x \in \mathfrak{g} \).
Main Lemma. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$.

If $\pi$ is an irreducible representation of $\mathfrak{g}$ on $V$, there exists a linear form $\lambda : \mathfrak{r} \to \mathbb{k}$ such that

$$\pi(x) = \lambda(x) I_v$$

for $x \in \mathfrak{r}$.

Remark. $\lambda([x,y]) I = \pi([x,y]) = \pi(x) \pi(y) - \pi(y) \pi(x) = (\lambda(x) \lambda(y) - \lambda(y) \lambda(x)) I_v = 0$ for $x, y \in \mathfrak{r}$.

$\Rightarrow \lambda|_{\mathfrak{r}} = 0$.

Corollary. Let $\mathfrak{g}$ be a solvable Lie algebra. Then any irreducible
representation of $\mathfrak{g}$ is one-dimensional.

The consequence of this result is the following theorem:

**Theorem (Lie's theorem)**

Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field $k$. Let $(\pi, V)$ be a representation of $\mathfrak{g}$ on a finite-dimensional vector space $V$ over $k$. Then there exists a basis of $V$ such that the matrices of $\pi(x)$, $x \in \mathfrak{g}$, are upper triangular.
Proof: By induction in $\dim V$.
Assume that $V$ is irreducible. By the first corollary, $\dim V = 1$ and $\pi(x) = \lambda(x) I_v$.
Assume that $\dim V > 1$. Then $\pi$ is not irreducible.
There exists an invariant subspace $U$, $0 \neq U \subseteq V$. Hence, $\dim U < \dim V$.
By induction assumption there exists a basis $e_1, \ldots, e_m$ of $U$ such that $\pi_u(x)$ are upper triangular.
\[ W = V / \mathcal{U} \quad \dim W < \dim V \]

There exist \( e_{m+1}, \ldots, e_m \) such that \( (e_{m+1} + \mathcal{U}, \ldots, e_m + \mathcal{U}) \) is a basis such that the rep. on the quotient is upper triangular.

\[
\pi(x) e_i = \sum_{j=1}^m a_{ij} e_j
\]

Proof of \( \oplus \): Let \( \pi(\gamma) = \mathcal{O}_L \), \( \pi(\delta) = p \). Then \( \mathcal{O}_L \) is a Lie subalg. of \( \mathfrak{L}(W) \) and \( p \) is a solvable ideal in \( \mathcal{O}_L \). Assume \( \pi \) is not \( \mathcal{O}_L \), then \( \pi(p) \neq 0 \). Then \( b = \mathcal{D}^p \pi p = 0 \). Then \( b = \mathcal{D}^p \pi p \) is an abelian ideal in \( \mathcal{O}_L \). Elements of \( \mathcal{O}_L \) form a commuting family of
linear transformations. Therefore they have a common eigenvector $v \neq 0$.

$$Tv = \lambda(T)v, \quad T \in B$$

$\lambda : B \to k$ is a linear form.

Let $S \in O_l$. Then $[S,T] \in B$ for all $T \in B$.

Claim: $\lambda([S,T]) = 0$.

Proof: Let $V_m$ be the subspace spanned by $v, Sv, \ldots, S^m v$.

Then

$$V_0 \leq V_1 \leq \ldots \leq V_m \leq \ldots$$

Since $V$ is finite-dimensional, it has to stabilize.
Assume that

\[ V_0 \supset V_1 \supset \ldots \supset V_m = V_{m+1} \]

Then \( V_m \) is \( \mathcal{S} \)-invariant.

Moreover, \( v_0, S v_0, \ldots, S^m v_0 \) is a basis of \( V_m \) \( \Rightarrow \) \( \dim V_m = m+1 \).

We claim that

\[ TS^m v - \lambda(t) S^m v \in V_{m-1} \]

for \( m = 0, \ldots, m \).

Clear for \( m = 0 \) by choice of \( \mathcal{S} \).