Cor. Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{L}(V)$ consisting of nilpotent linear transformations. Then $\mathfrak{g}$ is nilpotent.

Proof: $T \in \mathfrak{g}$, $T$ is nilpotent.

$\text{ad}T : \mathfrak{L}(V) \rightarrow \mathfrak{L}(V)$ is nilpotent

$\text{ad}T : \mathfrak{g} \rightarrow \mathfrak{g}$ is nilpotent.

$\Rightarrow \mathfrak{g}$ is nilpotent.

Let $\mathfrak{n}$ be a nilpotent Lie algebra

$\mathfrak{n} > E^1(\mathfrak{n}) > E^2(\mathfrak{n}) > \cdots > E^p(\mathfrak{n}) > 0$.

Pick a basis of $E^p(\mathfrak{n})$, extend to a basis of $E^{p-1}(\mathfrak{n})$, ...
In this basis the matrix of $\text{ad} x$ is

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

$\Rightarrow B(x,y) = tr(\text{ad}x, \text{ad}y) = 0$

Killing form vanishes on $n \times n$.

**Lemma.** The Killing form is zero on nilpotent Lie algebras.

**Example.**

$n = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ upper triangular matrices, zero on diagonal. $n$ is nilpotent. All subalgebras are nilpotent.
Three dimensional Lie algebras

K-algebraically closed field

$\mathfrak{g}_3$ - three dimensional Lie algebra

We proved that all Lie algebras in dimension 1 and 2 are solvable.

Let $\mathfrak{r}$ be the radical of $\mathfrak{g}_3$.

If $\mathfrak{g}_3 = \mathfrak{r}$, $\mathfrak{g}_3$ is solvable.

If $\mathfrak{r} \neq \mathfrak{g}_3$, $\mathfrak{g}_3/\mathfrak{r}$ is semisimple

hence $\dim(\mathfrak{g}_3/\mathfrak{r}) > 2$. This implies

that $\mathfrak{r} = \{0\}$; i.e. $\mathfrak{g}_3$ is semisimple.

Assume that $\mathfrak{g}_3$ is not solvable.

Then $\mathfrak{g}_3$ is semisimple.

Moreover, if $\mathfrak{r}$ is a nontrivial
ideal in \( \mathfrak{g} \), \( \dim \mathfrak{g} \leq 2 \)

and \( \mathfrak{g} \) is solvable and \( \mathfrak{g} \subset \mathfrak{h} \).

This is a contradiction.

So \( \mathfrak{g} \) has no nontrivial ideals.

**Def.** A Lie algebra is **simple** if it is not abelian and has no nontrivial ideals.

\( \mathfrak{g} \) is not abelian \( \implies \) \( \mathfrak{g} \) is simple.

Also \( \mathfrak{Dg} \neq \{0\} \) and \( \mathfrak{Dg} = \mathfrak{g} \)

\( \mathfrak{g} \ni x \mapsto \text{tr } \text{ad} x \) is a linear form.

It vanishes on \( \mathfrak{Dg} \).

\[
\text{tr } (\text{ad} [x,y]) = \text{tr } (\text{ad} x \text{ad} y - \text{ad} y \text{ad} x) = \\
= \text{tr } (\text{ad} x \text{ad} y) - \text{tr } (\text{ad} y \text{ad} x) = 0.
\]
Hence $x \mapsto \text{tr} \text{ad}_x$ vanishes on $\mathfrak{g}$. 

If all $\text{ad}_x$ are nilpotent, $\mathfrak{g}$ is nilpotent, and therefore solvable, contradicting our assumption.

There exists $x \in \mathfrak{g}$ which is not nilpotent. Hence it has an eigenvalue $\lambda \neq 0$.

$$(\text{ad}_x)x = 0 \implies 0$$

is an eigenvalue of $\text{ad}_x$.

Since $\text{tr} \text{ad}_x = \text{sum of eigenvalues}$, $\text{ad}_x$ has eigenvalues $\lambda, 0, -\lambda$ with multiplicity 1.
\[ \text{adj}(y) = \lambda y \implies \exists \ h \in \mathcal{O} \]
\[ (\text{adh})(e) = 2e \quad (h = \frac{2}{x}, \ e = y) \]
-2 is also an eigenvalue of \text{adh}.
\[ (\text{adh})(f) = -2f \]
e, f, h is a basis of \mathcal{O}.
\[ (\text{adh})[e,f] = [\text{adh}e,f] + [e, (\text{adh})f] = \]
\[ = 2[e,f] - 2[e,f] = 0. \]
\[ \implies [e,f] = \mu h \]
If \( \mu = 0 \), \[ [e,f] = 0 \] and \[ \text{Dg} = k \cdot e \oplus k \cdot f \]
contradicts \[ \text{Dg} = 0 \].
Hence \( \mu \neq 0 \). Scaling \( e \) or \( f \)
we can assume that
\[ [e,f] = h. \]
\[ [h_1e] = 2e \quad [h_1f] = -2f \]

\[ [e_1f] = h \]

\[ \mathfrak{sl}(2, k) \]

\[ h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \]

\[ [h, e] = [\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \]

\[ [h, f] = [\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \]

\[ [e, f] = [\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \]

\[ \mathfrak{g} = \mathfrak{sl}(2, k) \]

There exists only one three-dimensional simple Lie algebra over \( k \) - it is \( \mathfrak{sl}(2, k) \).
If \( \mathfrak{h} = \mathbb{R} \), \( \mathfrak{sl}(2, \mathbb{R}) = \{ [a, b] \mid a, b, c \in \mathbb{R} \} \) is a simple Lie algebra with basis

\[ h = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \]

\( \mathfrak{su}(2) = \{ [ia, b + ic] \mid a, e, c \in \mathbb{R} \} \)

\( -i \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = h \)

\( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = e - f, \quad i \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = i(e + f) \)

\( \operatorname{span}_c \mathfrak{su}(2) = \operatorname{span}_c \mathfrak{sl}(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{C}) \).