End \( (C^\infty(G)) \) asso. algebra with 1.

Any asso. algebra with 1 determines a Lie algebra by

\[
[A, B] = A\circ B - B\circ A.
\]

Let \( X, Y \) be two vector fields on \( G \). \( X, Y : C^\infty(G) \rightarrow \) satisfy Leibnitz rule

\[
X(f_1f_2) = f_1X(f_2) + X(f_1)f_2.
\]

\( X\circ X \) and \( Y\circ X \) are not vector fields, but \([X, Y] = [X_1, X_2] \) is (i.e. it satisfies Leibnitz).

Therefore all vector fields
on $G$ form a Lie algebra. This Lie algebra is infinite dimensional too big.
A vector field $X$ is left invariant if $X(F \circ \gamma(g)) = X(F) \circ \gamma(g)$ for all $g \in G$.

$$X(F)(g) = (X(F) \circ \gamma(g))(1) = X(F \circ \gamma(g)) = X, (F \circ \gamma(g))$$

Hence, $X$ is uniquely determined by $X,$. $\rightarrow$ Left invariant vector fields $\leftrightarrow T_{1}(G)$. 
\( L(G) \) all left invariant vector fields

\[
X, Y
\]

\[
Y(F \circ \gamma(g)) = Y(F) \circ \gamma(g)
\]

\[
X(Y(F \circ \gamma(g))) = X(Y(F) \circ \gamma(g)) = \]

\[
= X(Y(F)) \circ \gamma(g)
\]

\[
\Rightarrow [X, Y](F \circ \gamma(g)) = [X, Y](F) \circ \gamma(g)
\]

\[
\Rightarrow L(G) \text{ is a Lie subalgebra of } T(G) \text{ - finite dimensional.}
\]

\[
X_1 = \xi, \quad Y_1 = \eta
\]

\[
X(Y(F))_1 = \xi (Y(F))_1
\]
\[
Y(F)(g) = Y_1((F \circ \phi)(g)) = Y_1((F \circ m)(g, \cdot)) = \eta((F \circ m)(g, \cdot)) = (F \circ m)^{\eta}(g) \\
\Rightarrow \lambda(Y(F))(1) = \xi((F \circ m)^{\eta}) = (\xi \circ \eta)(F) \\
\Rightarrow [X, Y]_1(F) = (\xi \circ \eta)(F) - (\eta \circ \xi)(F) = [\xi, \eta](F).
\]
\( G \) — left invariant

\( G^{opp} \) — right invariant

\[ G \rightarrow G^{opp} \]

\[ \iota: g \rightarrow g^{-1} \]

is an isomorphism

\[ L(\iota): L(G) \rightarrow L(G^{opp}) \]

isomorphism of Lie algebras

\[ L(\iota) = -I \]

\[ L(G^{opp}) = L(G)^{opp} \]
Foliations

Let $M$ be a diff. manifold $i : L \to M$ an immersion such that

(a) $i : L \to M$ is a bijection,

(b) for any $m \in M$ there exists a chart $(U, \varphi, m)$, $p \geq \mathbb{Z}_+ \cap p + 2 = n$ $V \subseteq \mathbb{R}^p$, $W \subseteq \mathbb{R}^q$ open such that $\varphi(U) = V \times W$

$(\varphi \circ i)^{-1}(\{v\} \times W)$ is open in $L$ for any $v \in V$

$(\varphi \circ i) : (\varphi \circ i)^{-1}(\{v\} \times W) \to \{v\} \times W$ is a diffeo. for any $v \in V$

$(L, i)$ is a foliation of $M$. 


m ∈ M, \( i^{-1}(m) \subseteq L \) - connected component of \( L \) containing \( i^{-1}(m) \) is the leaf \( L_m \) through \( m \).

\( i : L_m \rightarrow M \) is an immersion.

\( L_m \) is not a submanifold.

\( m \mapsto \dim L_m \) is locally constant \( \Rightarrow \) constant on connected components of \( M \).

\( T(M) \) - tangent bundle of \( M \).

\[ \pi \] projection
Separability of leaves

A topological space is called separable if it has a countable basis of open sets.

Lemma. Let $G$ be a connected Lie group. Then $G$ is separable.

Proof. $U$ open neigh. of $\mathcal{L}$

- domain of a chart

$E C U$, $C$ countable (say points with rational coordinates in $\mathcal{L}$)

$G = \bigcup_{n=1}^{\infty} U^n$
\[ D = \bigcup_{m=1}^{\infty} C^m \]

\[ D \text{ is countable} \]

\[ g \in G \text{, } V \text{ neigh. of } g \]

\[ g \in V^m \implies g = g_1 \cdots g_n \quad g_i \in U \]

\[ \implies \exists V_i \text{ neigh. of } g_i \text{ such that } V_1 \cdots V_n < V. \quad V_i \ni c_i \in C \]

\[ \implies c_1 \cdots c_n \in V. \]

\[ D \text{ is dense in } G. \]

\[ \{ U_n \mid n \in \mathbb{N} \} \text{ fund. system of neigh. of } 1 \text{ in } G. \text{ We can assume that all } U_n \text{ are symmetric.} \]

\[ \text{Claim } \{ U_n \mid d \in D \} \text{ is a basis of topology of } G. \]
Let $V$ be open in $G$, $g \in V$. There exists $m \in \mathbb{N}$ such that $V^m g \subseteq V$. Since $D$ is dense in $G$, $V^m g \cap D \neq \emptyset$.

Let $d \in V^m g \cap D$

$V^m d c V^2 g c V$, $V^m$ is symmetric

d $\in V^m g \implies V^{m+d} \supseteq g$

$\implies g \in V^{m+d} \subseteq V$

$\implies V$ is union of elements of $\{V^m d ; m \in \mathbb{N}, d \in D\}$. 

We want to prove:

**Theorem.** Let $M$ be a
manifold such that all components of $M$ are separable. Let $(L,i)$ be a foliation of $M$. Then all leaves of $L$ are separable.

In particular, this implies that any foliation of a Lie group has separable leaves.

Lemma A. Let $M$ be a separable topological space. Let $(U_i; i \in I)$ be an open cover. Then $(U_i; i \in I)$ has a
countable subcover.

Proof. Let \( \{ V_j : j \in \mathbb{N} \} \) be a basis of topology of \( M \).

Every \( V_i \) is a union of elements of \( \{ V_j : j \in \mathbb{N} \} \). Let \( A \) be a subfamily of \( \{ V_j : j \in \mathbb{N} \} \) such that \( V_i \in A \implies V_i \subseteq U_i \)
for some \( i \in I \).

\[ m \in M \implies m \in V_m \subseteq U_i \]
\[ V_m \in A \quad A \text{ is a cover of } M. \]

For each \( V_m \) in \( A \) we pick \( U_{i(m)} \supseteq V_m \) \( (U_{i(m)}) \) is a subcover of \( (U_{i,j}) \). \( \square \)
Lemma B. Let $M$ be a connected topological space. Let $U = (U_i : i \in I)$ be an open cover of $M$ with following properties:

(i) all $U_i$ are separable,

(ii) $\{j \in I \mid U_i \cap U_j \neq \emptyset \}$ is countable for any $i \in I$.

Then $M$ is separable.

Proof. Let $U_{i_0} \neq \emptyset$. We say that $i \in I$ is accessible from $i_0$ in $m$ steps if there exist $i_1, i_2, \ldots, i_m \in I$ such that $U_{i_k} \cap U_{i_{k+1}} \neq \emptyset$ for $k = 0, \ldots, m-1$ and $i = i_m$. 
Let $A_n$ be the set of all indices in $I$ accessible in $n$ steps from $i_0$.

We claim that all $A_n$ are countable.

First, (ii) implies that for $A_i$, assume that $A_n$ is countable.

Let $j \in A_{n+1}$. Then there exists $i \in A_n$ such that $U_i \cap V_j \neq \emptyset$. Since $A_n$ is countable, by (ii) there exists countably many $j$ such that $U_i \cap V_j \neq \emptyset$ for $i \in A_n$. Thus $A_{n+1}$ is countable.
\[ A = \bigcup_{n=1}^{\infty} A_n \text{ is countable.} \]

Let \( V = \bigcup_{i \in A} V_i \). Then

\[ V \text{ is an open set in } M. \]

Let \( m \in V \). Then, \( m \in V_i \) for some \( i \). \( V_i \cap V \neq \emptyset \)

\[ \Rightarrow V_i \cap V_j \neq \emptyset \text{ for some } j \in A_m \Rightarrow i \in A_{m+1} \Rightarrow m \in U. \text{ Hence, } \overline{V} = V. \]

\[ \Rightarrow M \setminus \overline{V} \text{ is also open} \]

\[ M = V \cup (M \setminus \overline{V}). \text{ } M \text{ is connected} \Rightarrow M = U. \]

\( U_i \) are separable \( \Rightarrow M \text{ is separable. } \]
Lemma C. Let $M$ be a locally connected, connected topological space. Let $(U_{nj})_{n \in \mathbb{N}}$ be an open cover of $M$ such that each connected component of $U_n$ is separable. Then $M$ is separable.

Proof: Since $M$ is locally connected, the connected components of its open sets are open. $U_{nj, x : x \in A_n - \text{component of } U_n}$ if $U_{nj, x : x \in A_n, n \in \mathbb{N}}$ is an open cover of $M$. 
Let $A_{n, \alpha, j, m} = \{ \beta \in A_n | U_{n, \alpha} \cap U_{m, \beta} \neq \emptyset \}$.

Claim:

$A_{n, \alpha, j, m}$ is countable.

Since $U_{n, \alpha}$ is separable, $U_{n, \alpha} \cap U_{m}$ is open in $U_{n, \alpha}$ and can have only countably many components. We denote them by $S_{p}$. $S_{p}$ is open and connected, so it has to be in one unique component $U_{m, \beta(p)}$ of $U_{m}$.

Let $\beta \in A_{n, \alpha, j, m}$. Then
$\forall m, a \land \forall m, \beta \neq \emptyset$.

If $a \in \forall m, a \land \forall m, \beta$, $a \in \forall m, a \land \forall m$  
$\Rightarrow a$ is in a component of $\forall m, a \land \forall m$. Hence $a$ is in $S_p$ for some $p$ and $\beta = \beta(p)$. This proves the claim.

$\forall m, a$ are open and separable.
$\Rightarrow M$ is separable by Lemma B.
Let $M$ be a manifold and $(L, i)$ a foliation of $M$.
Assume that components of $M$ are separable.
Let $L$ be a leaf of $(L, i)$.
Since $L$ is connected, it is contained in a connected component of $M$. Hence, without any loss of generality we can assume that $M$ is connected.
Any point in $M$ has an open neighborhood $U$ such that
$\mathcal{U}$ is a domain of a chart $(\mathcal{U}, \varphi, \eta)$ such that $\varphi \circ g = \eta$ and $\varphi(\mathcal{U}) = \mathcal{V} \times \mathcal{W}$

$\mathcal{V} \subset \mathbb{R}^p$ connected open neigh. of $0 \in \mathbb{R}^p$

$\mathcal{W} \subset \mathbb{R}^q$ connected open neigh. of $0 \in \mathbb{R}^q$

$\varphi(m) = 0$

$(\varphi \circ i)^{-1}(\mathcal{V} \times \mathcal{W})$ is open in $\mathcal{L}$

$\varphi \circ i : (\varphi \circ i)^{-1}(\mathcal{V} \times \mathcal{W}) \to \mathcal{V} \times \mathcal{W}$ is a diffeomorphism for any $\omega \in \mathcal{U}$. 
for all $m \in M$ we get an open cover $(U_m : m \in M)$ of $M$. Since $M$ is separable by assumption, it has a countable subcover $(U_k : k \in \mathbb{N})$.

Pick a point $m_0 \in M$ and the leaf $L_0$ through $m_0$. Then $V_k = \varphi^{-1}(U_k) \cap L_0$ is a countable open cover of $L_0$. The open sets $(\varphi \circ i)^{-1}(\varnothing \times 3 \times W_k)$, $\sigma \in V_k$, are open and connected in $\mathcal{L}$. Hence, they are either in $L_0$ or disjoint of it.
It follows that components of $V_2$ are diffeomorphic to some $i_2 \times W_i$, i.e. they are separable. By lemma C, $L_0$ is separable.

This proves the theorem.

\[
\begin{array}{ccc}
N \xrightarrow{f} M & \text{Let } M \text{ be a manifold with separable components,} \\
N \xrightarrow{\bar{f}} L & \text{N a manifold and } f: N \rightarrow M \\
\end{array}
\]
a differentiable map.
Since \( i : L \rightarrow M \) is a bijection, there exists a map \( \overline{f} : N \rightarrow L \) such that the diagram commutes.

**Corollary.** Assume that \( \overline{f}(N) \) is contained in a union of countably many leaves. Then \( \overline{f} : N \rightarrow L \) is differentiable.

**Proof.** Let \( t \in N \), \( f(t) \in M \).

Find an open neighborhood of \( f(t) \) such that \( \gamma(U) = V \times W \), .....
Let \( O \) be a connected neigh. of \( t \) such that \( f(O) \subset U \), since the leaves of \( L \) are separable

\[
i^{-1}(U) \cap L = U(4) \cup \omega \times W
\]

over countably many \( v \in V \), for any leaf \( L_0 \). Therefore,

\[(p \circ f)(O) \subset C \times W\]

where \( C \) is a countable subset in \( V \).

Since \( O \) is connected, \n
\[(p \circ f)(O) \text{ is connected}\]
Hence, $C$ has to be connected. Since $C$ is countable, $C = \{ c_i \}$.

$$
\Rightarrow (f_0) (0) = \delta c \cap W
$$

diff. $(f_0) (\delta c \cap W)$ is open in $L_0$.

$$
\Rightarrow \exists \tilde{l}_0 \text{ is differentiable.}
$$

Lemma: $C$ countable subset in $\mathbb{R}^n$, $C$ connected $\Rightarrow C$ is a point.

Proof: Assume that $C$ is not a point. Then there exist $a, b \in C$

$a \neq b$. $a = (a_1, \ldots, a_p)$, $b = (b_1, \ldots, b_p)$

$a \neq b \Rightarrow a_i \neq b_i$ for some $i$. 
\( \exists \alpha \text{ between } a_i \text{ and } b_i \)
such that it is not \( i \)-th coordinate of any point in \( C \).

Put \( C_1 = \{ c \in C \mid c_i < \alpha \} \)
\( C_2 = \{ c \in C \mid c_i > \alpha \} \)

Then \( C_1 \neq \emptyset \), \( C_2 \neq \emptyset \), \( C = C_1 \cup C_2 \)
and \( C_1, C_2 \) are open in \( C \).

This contradicts the connectedness of \( C \). \( \square \)

Let \( M \) be a manifold with separable components. Let \((L, i) \) be a foliation of \( M \).
Assume that \( L \) has countably
many leaves. Then \( i: M \to L \) is differentiable, hence, \( M \cong L \) and \( L \) has only one leaf.

**Corollary**. If \( M \) has separable components, any nontrivial foliation has uncountably many leaves.
(L, i) foliation of M.

\[ \begin{align*}
T(L) & \text{ tangent bundle of } L \\
\downarrow & \\
T(i): T(L) & \rightarrow T(M) \\
\downarrow & \\
\downarrow & \\
L & \rightarrow M
\end{align*} \]

Can view \( T(i)T(L) \) as a vector subbundle of \( T(M) \).

A vector subbundle of \( T(M) \) is involutive if for any two vector fields \( X, Y \) on \( M \):

\[ X_m, Y_m \in E_m \Rightarrow [X, Y]_m \in E_m \]

Lemma: \( T(i)T(L) \) is an involutive vector subbundle
Converse is the Frobenius theorem.

Frobenius theorem: Let $E$ be an involutive vector subbundle of $T(M)$. An integral manifold $(N, j)$ of $E$ is

1. $N$ a diff. manifold
2. $j : N \to M$ is an injective immersion
3. $T_a(j) T_a(N) = E_{j(a)}$ for all $a \in N$. 

of $T(M)$. 

If \( m = \tilde{j}(s) \) we say that \((N, j)\) is an integral manifold through \( m \).

**Thm:** Let \( M \) be a diff.

manifold and \( E \) an involutive vector sub-bundle of \( T(M) \). Then there exists a foliation \((L, i)\) of \( M \) with the following properties:

(i) \((L, i)\) is an integral manifold for \( E \);

(ii) For any integral manifold \((N, j)\) of \( E \) there exists a unique differentiable map \( J : N \rightarrow L \).
such that the diagram
\[
\begin{array}{ccc}
N & \xrightarrow{f} & M \\
\downarrow{i} & & \downarrow{j} \\
\end{array}
\]

commutes and \( J(N) \) is an open submanifold of \( L \).

\((L, i)\) is unique

\((L, i)\) is the integral foliation of \( M \) with respect to \( E \).
$G$ Lie group

$\mathfrak{h} \subset \mathfrak{L}(G)$ Lie subalgebra

$E$ - vector subbundle of $T(G)$

$E_g = T(g \mathfrak{h})$

$\xi_1, \ldots, \xi_p$ basis of $\mathfrak{h}$

$X_1, \ldots, X_p$ left invariant vector fields such that $(X_i)_{g^{-1}} = \xi_i$.

$[\xi_i, \xi_j] = \sum_{k=1}^{p} c_{ijk}\xi_k$

$X_i, Y_j$ vector fields, $X, Y \in E_g$

$\Rightarrow \quad X = \sum_{i=1}^{p} f_i X_i, \quad Y = \sum_{j=1}^{p} g_j Y_j$

$[X,Y] = \sum_{i,j \neq i}^{p} \left(f_i X_i (g_j)X_j - g_j X_i (f_i)X_i \right)$

\[
+ \sum_{i,j \neq i}^{p} f_i g_j [X_i,X_j]
\]
\[= \sum_{i,j} (f_i X_i(g_j) X_j - g_j X_j(f_i) X_i) + \sum_{i,j,k} c_{ijk} f_i g_j X_k \]

\[\Rightarrow [X, Y]_g \in E_g\]

E is involutive.

(L, i) the integral foliation of G determined by E

(L, i) is the left foliation of G determined by E.

H is the leaf of L through \( l \in G \), \( g \in H \), \( \gamma(g) \cdot \iota : H \rightarrow G \)

is an integral manifold of E. Since H is a leaf then I
$\gamma(g) \cdot i : H \to G$ is an integral manifold thru $g \in H$.

$H$ is connected $\implies$ the integral manifold is open subset of $H$.

$\implies gH \subset H$.

$H$ is closed under multiplication

Apply to $g^{-1}$, $\delta(g^{-1}) \cdot i : H \to G$

integral manifold, containing $1$ since $g \in H$. Since $1 \in H$

$\implies g^{-1} \in H$. $H$ is a subgroup.

$H \times H \longrightarrow H$ $\mu$ is diff.

$\text{image is in}$
one leaf of $L$. Since components of $L$ are separable, hence leaves of $L$ are separable.

$\Rightarrow H \times H \xrightarrow{m} \Rightarrow H$ is differentiable.

$H$ is a Lie group.

Uniqueness

\[
\begin{array}{ccc}
H & \xrightarrow{i} & G \\
\cap & & \\
H' & \xrightarrow{i'} & G \\
\end{array}
\]

$L(i') : L(H') \rightarrow L(G)$ isomorphism onto $y$.

It follows that $T_y(i')(T_1(y(g))L(H'))$

$= T_1(\delta(g))(y) = E_g$
$H'$ is an integral manifold of left foliation attached to $y$ passing thru 1. Since $H'$ is connected $\alpha : H' \rightarrow H$

$H \xrightarrow{i} G$ diffeomorphism

$\uparrow \alpha \Downarrow \beta \xrightarrow{i'}$ on open set

$\alpha$ is homomorphism $\alpha(H') \cap H$

open subgroup $\Rightarrow \alpha(H') = H$. 
Tangent Lie algebra

Let $G$ be a Lie group and $H$ a subgroup of $G$.

$$\mathfrak{h} = \{ \xi \in \mathfrak{L}(G) \mid \exists I \subset \mathbb{R} \text{ interval } 0 \in I$$

$$\Gamma : I \to G \text{ diff.}$$

$$\Gamma_0(\mathbb{R}) (1) = \xi$$

$$\Gamma(I) \subset H \text{ tangent vector}$$

Lemma 1. $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{L}(G)$.

Proof. $\Gamma(t) = 1, t \in (-1, 1)$

$$\Gamma_0(\mathbb{R})(1) = 0$$

$$\Gamma(I) \subset \mathfrak{h}$$

$$\Rightarrow \ 0 \in \mathfrak{h}.$$
\[
\begin{align*}
\Gamma_1 : I_1 \to G, \quad \Gamma_2 : I_2 \to G \\
\mathcal{I} = I_1 \cap I_2, \quad T_\mathcal{I}(\Gamma_1) = \xi_1, \quad T_\mathcal{I}(\Gamma_2) = \xi_2 \\
\Gamma(t) = \Gamma_1(t) \Gamma_2(t), \quad t \in \mathcal{I}, \quad \Gamma(I) \subset \mathcal{I} \\
T_\mathcal{I}(\Gamma(t)) = T_{\mathcal{I}(m)}(T_\mathcal{I}(\Gamma_1)(1), T_\mathcal{I}(\Gamma_2)(1)) \\
= T_\mathcal{I}(\Gamma_1)(1) + T_\mathcal{I}(\Gamma_2)(1) = \xi_1 + \xi_2 \\
\Rightarrow \xi_1 + \xi_2 \in \mathcal{I}^1.
\end{align*}
\]

\[
\begin{align*}
\lambda \in \mathbb{R}, \quad \xi_\lambda(t) &= \Gamma_1(\lambda t) \\
T_\mathcal{I}(\xi_\lambda)(1) &= T_\mathcal{I}(\Gamma_1)(\lambda) = \lambda \xi_1 \\
\Rightarrow \lambda \xi_1 \in \mathcal{I}^1.
\end{align*}
\]

\[\mathcal{I}^1 \text{ is a vector space.}\]

\[
\begin{align*}
\forall \lambda \in \mathbb{H}, \quad \Gamma_1(t) = \lambda \Gamma_1(t) \lambda^{-1} \in \mathcal{H}
\end{align*}
\]
To \( T_0(\Gamma_h)(1) = Ad(h) T_0(\Gamma)(1) = Ad(h) \xi \),

\[ \Rightarrow Ad(h)(\eta) \in \mathfrak{h}. \]

Let \( \eta \in \mathfrak{h}, \exists \Gamma_t: I \rightarrow H, \)

\[ \eta = T_0(\Gamma_t)(1) \quad Ad(\Gamma_t(t))(\eta) \in \mathfrak{h}, t \in \Sigma \]

differentiate with respect to \( t \in I \).

\[ \Rightarrow [\eta, \gamma] \in \mathfrak{h} \quad \gamma \in \mathfrak{h} \]

is a Lie subalgebra of \( L(G) \).

\[ \mathfrak{h} \text{ is the tangent Lie algebra of } H. \]