A topological space $X$ is a Baire space if the intersection of any countable family $\{U_n, n \in \mathbb{N}\}$ of open dense sets $U_n$ is dense in $X$ (i.e., $\bigcap_{n \in \mathbb{N}} U_n$ is dense in $X$).

**Lemma (Category Theorem)**

A locally compact space $X$ is a Baire space.

**Proof.** Let $(U_n, n \in \mathbb{N})$ be a family of open dense subsets in $X$. Let $V = V_i$ be a nonempty open set in $X$ such that $V_i$ is compact. (X is locally compact).
Consider, $V_1 \cap U_1$ - this is a nonempty open set in $X$ since $U_1$ is dense in $X$.

Therefore, there exists a nonempty open set $V_2$ such that $V_2 \subset V_1 \cap U_1$,

and $\overline{V_2}$ is compact (since $X$ is locally compact).

Continue inductively:

$$\Rightarrow V_{m+1} \subset \overline{V_m} \subset V_m \cap U_m$$

Since these are compact sets,

$$\Rightarrow \bigcap_{m \in \mathbb{N}} \overline{V_m} = W$$
compact and nonempty. This implies that $W \subset \cup_m$ for all $m \in \mathbb{N}$, hence $W \subset \bigcap_{m \in \mathbb{N}} U_m$.

$\Rightarrow \bigcap_{m \in \mathbb{N}} U_m \neq \emptyset \Rightarrow \bigcap_{m \in \mathbb{N}} U_m$ is dense in $X$. \(\Box\)

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**Manifolds are Baire spaces!**

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A locally compact space $X$ is countable at infinity if it is a union of countably many compact sets.
Theorem (open mapping).
Let $G$ be a locally compact group countable at infinity. Assume that $G$ acts continuously on a compact space $M$, and that the action is transitive. Then the orbit map $\omega_m : G \to M$ (given by $g \mapsto g \cdot m$) is open.

Lemma. Let $G$ be a Lie group. Then, the following are equivalent:
(i) $G$ is countable at infinity
(ii) $G$ has countably many components.

**Proof.** (i) $\Rightarrow$ (ii) Assume that $G$ is countable at $\infty$, $\Rightarrow$

$$G = \bigcup_{n \in \mathbb{N}} K_n$$

where $K_n$ are compact $(G_i, j \in I)$ are open and disjoint $(G_i \cap K_n, j \in I)$

is an open cover of $K_m$. It has a finite subcover $\Rightarrow$

$K_m$ intersects finitely many $G_i, i \in I$. $\Rightarrow$

Union of $K_m = G$ intersect countably many $G_i \Rightarrow I$ is countable.
(ii) $\Rightarrow$ (i) $G$ has countably many components.
It is enough to show that $G_0$ is a union of countably many compact sets.
This follows from the lemma.
Lemma. Let $G$ be a connected Lie group. Let $U$ be a neighbor.
of 1 in $G$. Then $G = U \bigcup_{n \in \mathbb{N}} U^n$.

Proof. Can assume that $U$ is symmetric. Then $H = U \bigcup_{n \in \mathbb{N}} U^n$ is a subgroup.
$H$ is also open. $\Rightarrow$
h \in U^n \text{ for some } n \in \mathbb{N}

\bigcirc h \cdot U \subset U^{n+1}

\text{neighborhood of } h \Rightarrow h \cdot U \subset H

H \text{ is open } \Rightarrow G \text{ is union of } H \text{ cosets } = \text{ open. } G

\text{is connected } \Rightarrow H = G. 

G \text{ is a Lie group with countably connected components, } G \text{ is countable at } \infty.

G \text{ acts differentiably on manifold } M. \text{ If the action is transitive, for } m \in M,

\forall m : G \to M \text{ is open.}
The orbit map $\omega_m : G \to M$ has to have constant rank (subimmersion). It is open only if the rank is maximal, i.e., $\omega_m$ is a submersion.

Proof of open mapping theorem:
Let $U$ be an open neighborhood of $1 \in G$. We claim first that $\omega_m(U)$ is a neighborhood of $m$. Assume that $V$ is a compact symmetric neighborhood of $1$. 

such that $V^2 \subset U$.

**Existence:** multi. is continuous.

Let $V_1 = 1$ open such that $V_1^2 \subset U$.

Can shrink it to $V_2 \in 1$ which is a neigh. of 1 and compact.

$V_2 \subset U$, $V_2 \cap V_2^{-1}$ is compact and neigh. of 1 ($V_2$ is a neigh. and $V_2^{-1}$ is a neigh.)

$\Rightarrow V = V_2 \cap V_2^{-1}$ satisfies our assumption.

$(v, U)$ $v \in G$ is a cover of $G$.

Since $G = \bigcup_{n=1}^{\infty} U_n$
\((g \cdot \text{int}(V); g \in G)\) is a cover of \(K_n \implies \exists \text{ finite subcover} (g_m \cdot \text{int}(V); m \in N)\) is an open cover of \(G\).

\((g_m \cdot V; m \in N)\) is a cover of \(G\).

\[ V_m = M - \omega(g_m \cdot V) = \]
\[ = M - g_m \cdot V \cdot m \]

\(\uparrow\)

compact \implies \text{closed}

\(U_m\) is open

\[ \bigcap_{\infty} U_m = \bigcap_{m=1}^{\infty} (M - g_m \cdot V \cdot m) = \]
\[ = M - \bigcup_{m=1}^{\infty} g_m \cdot V \cdot m = M - G \cdot M = \]
\[ = \emptyset.\]
Since $M$ is a Baire space, at least one $U_m$ is not dense in $M$.

$$U_m = M - g_m(V, m) \text{ not dense}$$

$$\Rightarrow g_m(V, m) \text{ has nonempty interior}$$

$V, m$ has nonempty interior

$V, m$ is a neigh of $g_m$

$$\Rightarrow g_m^{-1}(V, m) \text{ is a neigh of } m$$

$g_m^{-1}(V, m) \subset V^0, m \subset U, m$

$$\Rightarrow U, m \text{ is a neigh of } m.$$ Can complete the proof.

$O$ open in $G$, $g \in O$. 
$g^{-1}O$ is a neigh. of 1
$g^{-1}O \cap m$ in a neigh. of $m$

$\Rightarrow O \cap m$ is a neigh. of $g \cdot m$.
$\Rightarrow O \cdot m$ is open!
Universal covering spaces

\( X \) a manifold, \( x_0 \) - base point

\( X \sim (\tilde{X}, \tilde{x}_0) \) is a universal

\( \downarrow \quad \text{covering space} \)

\( \downarrow \quad X \sim x_0 \) if \( X \) is a connected manifold such that \( p: \tilde{X} \rightarrow X \) is a covering \( p(\tilde{x}_0) = x_0 \), and for any other covering space \((Y, y_0)\)

\[ X \xleftarrow{\sim} \text{unique} \]

\( \downarrow \quad X \xrightarrow{p} \sim \quad Y \)

\( \downarrow \quad \text{must} \)

\( X \xrightarrow{\sim} \text{be identity} \)

\( \Rightarrow\ \)

Another universal cover must be diffeo.
Universal covering space is unique up to an isomorphism.

Universal covering spaces are simply connected, i.e.
\[ \pi_1(\tilde{X}, \tilde{x}_0) = \{ 1 \} \]
\( \tilde{x} \sigma \in p^{-1}(x_0) \) \( (\tilde{x}, \sigma) \) is a universal covering space
\( \exists \) unique \( T_\sigma : \tilde{X} \to \tilde{X} \)
\( \tilde{x} T_\sigma \tilde{x} = T_\sigma (x_0) = x_0 \)
map \( \sigma - \) deck transformation
\( \tilde{X} \xrightarrow{T_\sigma} \tilde{X} \quad T_\sigma (p^{-1}(x_0)) = p^{-1}(x_0) \)
Deck transformations form a group \( \equiv \pi_1(X, x_0) \)
\(G\) a connected Lie group

\((\tilde{G},\tilde{1})\) universal covering

A unique Lie group structure on \(G\) (compatible with manifold structure on \(\tilde{G}\)) such that

\(\tilde{1}\) is the identity and

\(p : \tilde{G} \rightarrow G\) is a Lie group morphism.

\(\tilde{G}\) - universal covering group.

Lifting property

\((X,x_0)\) connected manifold
\((Y, p, y_0)\) a covering space

\[F : (Z, z_0) \to (Y, y_0)\]

\((Z, z_0)\) connected and simply connected manifold, \(F\) diff. map.

Then there exists the unique diff. map \(F' : Z \to Y\) such that \(F'(z_0) = y_0\).

Construction of the group structure on \(G\).
\( \tilde{G} \times \tilde{G} \) is simply connected

\[
(\tilde{G} \times \tilde{G}, 1 \times 1) \xrightarrow{\tilde{m}} (\tilde{G}, 1)
\]

\( \tilde{m} \) is unique.

\( \tilde{m} : G \times G \rightarrow \tilde{G} \) is a diff. map, - binary operation. Have to show that it defines a group structure.

We have

\[
p \circ \tilde{m} = m \circ (p \times p).
\]