Let $A$ be a finite dimensional associative algebra over $R$ with identity $1$.

$A$ is a finite-dimensional vector space - manifold.

Multiplication $m: A \times A \rightarrow A$

$m(x, a_1, \ldots, x_{m,n}, y, a_1, \ldots, y_{m,n}) = \\
= \sum_{j=1}^{m} x_j y_j a_i \cdot a_j = \sum_{i,j,k=1}^{m} x_i y_i C_{ijk} a_k$.

$a_i \cdot a_j = \sum_{k=1}^{m} C_{ijk} a_k$.

$m$ is differentiable

$G$ - group of regular elements

in $A$ is an open submanifold - Lie group.
$a \mapsto a^t$ is an involution on $A$ if it is linear

(i) $(a^t)^t = a$

(ii) $(a, b)^t = b^t a^t$

$F = 1^t, F^t = (1^t)^t = (1^t (1^t)^t)^t = (1^t 1^t)^t = 1^t 1^t = (1^t)^t = 1$

Examples

1. $A = M_n(\mathbb{R})$, $A^* \text{ is an involution }$

2. $A = M_n(\mathbb{C})$, $A^{\dagger}$ conjugate transpose
\[
H = \{ g \in A \mid g^* g = g g^* = 1 \}
\]

\[ \Rightarrow H \triangleleft G \]

\( H \) is a subgroup

\[ g, h \in H \]

\[
(gh)(gh)^* = gh h^* g^* = 1
\]

\[
(gh)^* gh = h^* g^* g h = 1
\]

\[
(g^*)(g^*)^* = g^* g = 1, \quad g \in H
\]

\[
(g^*)^* g^* = g g^* = 1 \quad \Rightarrow g^* \in H
\]

Example ①  \( A = M_n(\mathbb{R}), \quad H = O(n) \) orthogonol matrices

②  \( A = M_n(\mathbb{C}), \quad H = U(n) \) unitary matrices
\( \psi : A \rightarrow S \) is a submersion at 1, \( \mathcal{U} \) open neighborhood of 1 in \( A \) such that \( H \cap \mathcal{U} = (\psi|_\mathcal{U})^{-1}(1) \) and \( \psi|_\mathcal{U} \) is a submersion.

Hence

\[
T_1(H) = \ker T_1(\psi)
\]

\[
\Rightarrow T_1(H) = \{ \alpha \in A \mid \alpha^2 = -\alpha \}
\]

**Example:** \( \mathcal{O}_A = M_m(\mathbb{R}) \)

\( H = \mathcal{O}(m) \), \( T_1(H) = \left\{ T \in M_m(\mathbb{R}) \mid T = -T^T \right\} \)

antisymmetric matrices

\[
\dim T_1(H) = \frac{m(m-1)}{2}
\]

\[
\dim \mathcal{O}(m) = \frac{m(m-1)}{2}
\]
2. \[ A = M_n(\mathbb{C}), \quad T^* = T^\dagger \] hermitian adjoint

\[ H = U(n) = \text{unitary matrices} \]

\[ T_1(H) = \{ T \in M_n(\mathbb{C}) \mid T^* = -T \} \]

\[ \dim T_1(H) = \frac{n^2 - n}{2}, \quad 2 + n = n^2 \]

\[ \dim U(n) = n^2 \]

pure imaginary diagonal

complex off diagonal coefficients

\[ \det : M_n(\mathbb{R}) \to \mathbb{R} \]

\[ \det : O(n) \to \mathbb{R}^* \]

morphism of Lie groups
\[
\det(TT^*) = \det(T) \cdot \det(T^*) = (\det T)^2 = 1
\]

\[
\Rightarrow \det T = \pm 1.
\]

\[
\det : O(m) \rightarrow \{\pm 1\}
\]

\[
SO(m) = \{T \in O(m) \mid \det T = 1\}
\]

**special orthogonal group**

Let \(O(m)_0\) be the identity component of \(O(m)\). Then \(\det(O(m)_0)\) is a connected subset of \(\{\pm 1\}\) which contains 1 \(\Rightarrow\) it is equal to 1.

It follows that \(O(m)_0 \subset SO(m)\).

\(SO(m)\) is a union of components of \(O(m)\) - open subgroup of \(O(m)\).
$$\dim \text{SO}(n) = \frac{n(n-1)}{2}$$

$$\text{SO}(2), \quad \dim \text{SO}(2) = 1$$

$$T \in \text{SO}(2)$$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1$$

$$T^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$T^* = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow a = d, \quad b = -c$$

$$T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$\text{SO}(2) \sim \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\}$$

$$\text{SO}(2)$$ is a circle.

$$\text{SO}(2)$$ is connected

$$\pi_1(\text{SO}(2)) = \mathbb{Z}.$$
$O(n)$ is a closed subset of $M_n(\mathbb{R})$

Moreover, since columns of an orthogonal matrix have norm 1, the coefficients are in $[-1,1]$. $O(n)$ is bounded in $M_n(\mathbb{R})$.

$\Rightarrow O(n)$ is a compact Lie group. It has a finite number of components.

$SO(n)$ is also a compact Lie group.

$SO(n)$, $n \geq 2$, is connected.

We know this for $n = 2$. $GL(n,\mathbb{R})$ acts on $\mathbb{R}^n$. 
Consider the orbit of \((1, 0)\).

The differential of the orbit map at \(I\) is

\[
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots \\
a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & \vdots \\
a_{mj} & \cdots & a_{mn}
\end{pmatrix} \rightarrow 
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots \\
a_{ii} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & \vdots \\
a_{mj} & \cdots & a_{nn}
\end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 
\begin{pmatrix}
\vdots \\
a_{i1} \\
\vdots \\
a_{m1}
\end{pmatrix}
\]

Restrict the action to \(SO(m)\).

Then the orbit of \((1, 0)\) is

the sphere \(S^{m-1}\) in \(\mathbb{R}^m\) submanifold

\[
\{(x_1, \ldots, x_m) \mid x_1^2 + \ldots + x_m^2 = 1\}
\]

\[
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots \\
a_{ii} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & \vdots \\
a_{mj} & \cdots & a_{nn}
\end{pmatrix} \rightarrow 
\begin{pmatrix}
\vdots \\
a_{i1} \\
\vdots \\
a_{m1}
\end{pmatrix}
\]

Therefore, \(SO(m)\) act transitively

on \(S^{n}\). \((1, 0)\) "north pole" of the sphere,
The differential of the orbit map is

\[ T_1(w) : T_1(SO(n)) \rightarrow T_{ij}(\mathbb{R}^m) \]

antisymmetric matrices

\[
\begin{pmatrix}
0 & & \\
\vdots & \ddots & \\
& \ddots & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 \\
a_{21} \\
\vdots \\
a_{m1}
\end{pmatrix} \in \mathbb{R}^m
\]

\[ T_1(w) \text{ is surjective.} \]

\[ w \text{ is a submersion.} \]

\[ \Rightarrow w \text{ is an open map.} \]

The orbit of the identity component \( G_0 \) of \( SO(n) \)

is open. Since \( SO(n) \) is compact, \( G_0 \) is also compact, the orbit is also closed.
Since $S^{n-1}$ is connected the $G_0$-orbit of the pole is equal to $S^{n-1}$. Hence the action is transitive!

Stabilizer of the pole

\[
\begin{pmatrix}
0 & 1 & \cdots & 0 \\
-1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\text{upper diagonal} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots \\
\text{lower diagonal} & \cdots & \cdots & \text{1}
\end{pmatrix}
\]

orthogonal \quad \Rightarrow \quad A

\text{det } T = \text{det } A \Rightarrow \text{det } A = 1

\Rightarrow A \in SO(m-1),

The above map \ is a Lie group morphism of \( SO(m-1) \) into \( SO(n) \).

Proof of connectedness.
Assume that $SO(n-1)$ is connected. Then $S$ is connected. Hence it is contained in $G_0$, the identity component of $SO(n)$. We proved that $G_0$ acts transitively on $S^{n-1}$. Let $T \in SO(n)$, then $w(T) \in S^{n-1}$.

By transitivity, there exists $T_0 \in G_0$ such that $w(T) = w(T_0)$. This implies that $w(T_0^{-1}T) = \text{pole}$.

Hence, $T_0^{-1}T \in S$. By induction assumption $S \subseteq G_0 \Rightarrow T \in G_0 \Rightarrow G_0 = SO(n)$.
Since $T \in \mathcal{U}(n)$ is unitary, its columns are vectors of norm 1 in $\mathbb{C}^n$. \(\Rightarrow\) Matrix coefficients are $\leq 1$ in absolute value.

$\mathcal{U}(n)$ is closed and bounded in $M_n(\mathbb{C})$.

$\Rightarrow \mathcal{U}(n)$ is a compact Lie group.

$T \in \mathcal{U}(n)$, \(TT^* = I\)

$\det(TT^*) = \det(T) \cdot \det(T^*) = \det(T) \cdot \overline{\det(T)} = (\det(T))^2 = 1$

$\Rightarrow |\det T| = 1$. 
Hence

$$\det: \mathbb{M}_n(\mathbb{C}) \xrightarrow{\text{diff}} \mathbb{C}$$

$$\uparrow$$

$$\mathbb{G}L(n, \mathbb{C}) \xrightarrow{\text{diff}} \mathbb{C}^*$$

$$\uparrow$$

$$\det: \mathbb{U}(n) \xrightarrow{\text{diff}} \{z \mid |z| = 1\}$$

Hence $\det: \mathbb{U}(n) \xrightarrow{\text{diff}} \{z \mid |z| = 1\}$ is a Lie group morphism.

$\det$ is a submersion $\Rightarrow$

rank is either 0 or 1

If rank = 0, $T_s(\det) = 0$ for all $s$, $\det$ is locally constant on $\mathbb{U}(n)$, constant on the identity component. $\Rightarrow$ Constant on each component.
Since $U(n)$ has finitely many components, $\text{im} \det$ would be a finite subgroup of $\{z | |z| = 1\}$.

\[
\begin{pmatrix}
 z \\
 0 \\
 \vdots \\
 0 \\
 1
\end{pmatrix}, |z| = 1 \text{ is in } U(n)
\]

\[
\det \begin{pmatrix}
 z \\
 0 \\
 \vdots \\
 0 \\
 1
\end{pmatrix} = z
\]

Contradiction.

Hence rank is $1$. It follows that $\det$ is a submersion.

$\det: U(n) \to \{z | |z| = 1\}$ is surjective.

$SU(n) = \{T \in U(n) | \det T = 1\}$

special unitary group
\[ \text{dim } SU(n) = n^2 - 1. \]

\[ \text{SU}(2) \quad \text{dim } SU(2) = 3 \]

\[ T = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \alpha \bar{\beta} - \beta \bar{\alpha} = 1 \]

\[ T^{-1} = \begin{pmatrix} \overline{\alpha} & -\beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = T^{\ast} = \begin{pmatrix} \overline{\alpha} & \beta \\ \beta & \bar{\alpha} \end{pmatrix} \]

\[ \Rightarrow \delta = \overline{\alpha}, \quad \gamma = -\bar{\beta} \]

\[ T = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1 \]

\[ \alpha = \text{Re} \alpha + i \text{Im} \alpha \quad \beta = \text{Re} \beta + i \text{Im} \beta \]

\[ (\text{Re} \alpha)^2 + (\text{Im} \alpha)^2 + (\text{Re} \beta)^2 + (\text{Im} \beta)^2 = 1 \]

\[ \text{SU}(2) \text{ is a 3-dim sphere in } \mathbb{R}^4 \]

\[ \text{SU}(2) \] is connected and simply connected!
$SU(2)$

$$\begin{pmatrix} x & y+iz \\ y-iz & -x \end{pmatrix}$$

self adjoint

complex matrices

$\mathbb{C}^2$ acts on $\mathbb{SL}$

$\mathbb{R}^3 \rightarrow \mathbb{SL}$

linear isomorphism

$$\text{det} \, \varphi(x, y, z) = \begin{vmatrix} x & y+iz \\ y-iz & -x \end{vmatrix} = -(x^2 + y^2 + z^2)$$

$SU(2)$ acts on $\mathbb{SL}$

$$(\mathbb{T})^A \rightarrow \mathbb{T} \mathbb{A} \mathbb{T}^*$$
\[
\begin{align*}
(TAT^*) &= (TA)^*A^T T^* = \\
&= TAT^* + r(TAT^*) = r(T^*TA) = \\
&= r(A) = 0
\end{align*}
\]

\[m(T, A) \in Y \mathcal{E} \]

\[
SU(3) \times Y \mathcal{E} \rightarrow Y \mathcal{E}
\]

\[
(ST)A(ST)^* = STAT^*S^* = \\
= m(T, A)S^* = m(S, m(T, A)) = \\
= m(ST, A)
\]

\[
\mu : SU(2) \times Y \mathcal{E} \rightarrow Y \mathcal{E}
\]

is a diffusion action of \( SU(2) \) on \( Y \mathcal{E} \).

The action of \( SU(2) \) on \( Y \mathcal{E} \) is linear. Defines a
Linear action on $T \mathbb{R}^3$

$\Rightarrow$ defines a homomorphism $p : SU(2) \rightarrow O(3)$

Since

$$\det(TAT^*) = \det(T) \det A,$$

$$\det(T^*) = \det(TT^*) \det A = \pm$$

$$= \det(A)$$

$SU(2)$ is connected — the image is in the identity component of $O(3)$, inside $SO(3)$

$p : SU(2) \rightarrow SO(3)$

$^\top$

3 dim $\quad$ 3 dim
\[
\begin{align*}
\left(\begin{array}{c}
\alpha \\ -\beta
\end{array}\right) & \left(\begin{array}{c}
x \\
y + iz
\end{array}\right) \\
\left(\begin{array}{c}
y - iz \\
x
\end{array}\right) & \left(\begin{array}{c}
\alpha \\ -\beta
\end{array}\right) \\
\left(\begin{array}{c}
-\beta x \\
\bar{\alpha} (y - iz)
\end{array}\right) & \left(\begin{array}{c}
\alpha (y + iz) \\
-\beta x
\end{array}\right)
\end{align*}
\]

\[= \left(\begin{array}{c}
\alpha x + \beta (y - iz) \\
-\beta x + \alpha (y - iz)
\end{array}\right) \left(\begin{array}{c}
\alpha \\ -\beta
\end{array}\right) + \left(\begin{array}{c}
\alpha x + \beta (y - iz) \\
-\beta x + \alpha (y - iz)
\end{array}\right) \left(\begin{array}{c}
\alpha \\ -\beta
\end{array}\right)
\]

\[= \left(\begin{array}{c}
\alpha x + \beta (y - iz) + \alpha \beta (y + iz) - |\beta|^2 x \\
-\beta x + \alpha (y - iz) - \beta^2 (y + iz) - \alpha \beta x
\end{array}\right) \left(\begin{array}{c}
\alpha \\ -\beta
\end{array}\right)
\]

\[= \left(\begin{array}{c}
|\alpha|^2 - |\beta|^2 x + 2 \text{Re}(\alpha \overline{\beta} (y + iz)) \\
-2 \alpha \overline{\beta} x + (\alpha^2 - |\beta|^2) y - i(\alpha^2 + |\beta|^2) z
\end{array}\right)
\]

\[= \left(\begin{array}{c}
|\alpha|^2 - |\beta|^2 x - 2 \text{Re}(\alpha \overline{\beta} (y + iz)) \\
-2 \alpha \overline{\beta} x + (\alpha^2 - |\beta|^2) y + i(\beta^2 + \alpha^2) z
\end{array}\right)
\]

\[= \left(\begin{array}{c}
|\alpha|^2 - |\beta|^2 x - 2 \text{Re}(\alpha \overline{\beta} (y + iz)) \\
-2 \alpha \overline{\beta} x + (\alpha^2 - |\beta|^2) y - i(\beta^2 + \alpha^2) z
\end{array}\right)
\]
\[
\begin{pmatrix}
|\alpha| - |\beta| & 2 \text{Re}(\alpha \beta) - 2 \text{Im}(\alpha \beta) \\
-2 \text{Re}(\alpha \beta) & \text{Re}(\alpha^2 - \beta^2) - \text{Im}(\alpha^2 + \beta^2) \\
-2 \text{Im}(\alpha \beta) & \text{Im}(\alpha^2 - \beta^2) - \text{Re}(\alpha^2 + \beta^2)
\end{pmatrix}
\]

\[\ker p = \left\{ \left( \frac{\alpha}{\beta}, \frac{\beta}{\alpha} \right) : \text{such that} \right.\]

\[1|\alpha|^2 - |\beta|^2 = 1, \quad 1|\alpha|^2 + |\beta|^2 = 1\]

\[\Rightarrow 2|\alpha|^2 = 2 |\alpha|^2 = 1 \quad \boxed{|\alpha| = 1}\]

\[|\beta|^2 = 0, \quad |\beta| = 0 \quad \beta = 0\]

\[\text{Re} \alpha^2 = 1, \quad \text{Im} \alpha^2 = 0\]

\[\alpha^2 = 1, \quad \alpha = \pm 1\]

\[\ker p = \{ I, -I \} \]
\[ p \text{ is a differentiable homomorphism} \]
\[ \Rightarrow p \text{ is a Lie group morphism} \]
\[ p \text{ is a subimmersion} \]
\[ \text{since ker } p \text{ is discrete} \]
\[ p \text{ must be an immersion} \]
\[ T_i(p) : T_i(SU(2)) \rightarrow T_i(SO(3)) \]
\[ \text{injective} \quad \text{3-dim} \quad \text{3-dim} \]
\[ \Rightarrow p \text{ is a local diffeomorphism} \]
\[ \Rightarrow p \text{ is an open map} \]
\[ p(SU(2)) \text{ is an open subgroup of } SO(3) \]
\[ \text{Since it is compact, it is closed. Since } SO(3) \text{ is connected, } p(SU(2)) = SO(3)! \]
Covering groups

Let $\pi: p$ be a covering map

if $X$ and $Y$ are connected

$p$ is surjective, any yet

has an open connected neigh. $U$ such that $p^{-1}(U)$ is a union

of connected components $O_i$ and $p|_{O_i}: O_i \to U$ is a
diffeomorphism.

Let $G$ and $H$ be two

connected Lie groups

$p: G \to H$ is a

Lie group morphism.
Theorem: Equivalent:

(i) \( p \) is a covering map;
(ii) \( T_1(p): T_1(G) \to T_1(N) \) is an isomorphism.

Proof. (ii) is equivalent with \( p: G \to H \) being a local diffeo \((p \) is a submersion in general).

(i) \(\Rightarrow\) (ii) \( p \) is a covering map \(\Rightarrow p \) is a local diffeo. at 1 \(\Rightarrow T_1(p) \) is a linear isomorphism.

(ii) \(\Rightarrow\) (i), \( p \) is a local diffeo.

\( p \) is open, \(\Rightarrow p(G) \) is an open subgroup of \( H \).
All $p(G)$-cosets in $H$ are open.
Therefore, they are closed too

$\Rightarrow p$ is surjective since $H$ is connected

$kep p = D$ is a closed normal subgroup of $G$

Since $p$ is a local diffeo, $EU_1$

open neighborhood of $1$ such that $VMD = f(g)$.

and $p|_0 : 0 \rightarrow p(U)$ is a diffeomorphism.

Let $d<U D \cap d(U) = d(DU) = \{d\}$

$\Rightarrow D$ is discrete subgroups.

Lemma: $D$ is a central subgroup of $G$.

$G \times D \longrightarrow D \quad (g,d) \rightarrow (gdg^{-1})$

Continuous. Image of
$G \times \{ d \}$ is connected $\Rightarrow$ a point $(1, d) \rightarrow d \Rightarrow$ the image in $\{d\}$, $gdg^{-1} = d \Rightarrow$ $gd = dg$

$\Rightarrow d$ is in the center of $G$.  

Back to the proof.  

$V$ open and connected $\uparrow$

such that $V^2 \subset V$, $p|_V : V \rightarrow p(V) = 0$

is a diffeo.

$p^{-1}(V) = \{ g \in G \mid p(g) = p(h), h \in V \}^2$

$\Rightarrow d = g^{-1}h \in D$

$gd = h$

$\Rightarrow g \in D \cdot V$

$p^{-1}(V) = D \cdot V$
\[ d_1 V \cap d_2 V \neq \emptyset \]

\[ \implies d_1 V_1 = d_2 V_2 \implies d_1 d_2 = v_2 v_1^{-1} \]

\[ D \quad V^2 < U \]

\[ \implies d_1^{-1} d_2 = 1 \implies d_1 = d_2 \]

\[(d_1 V \; \text{and} \; d_2 D) \text{ are components of } p^{-1}(V).\]

\[ G \text{ is a covering space.} \]

\[ \text{If } p: G \to H \text{ is a covering} \]
Quaternions and symplectic groups

\[ A = \left\{ \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) \mid \alpha, \beta \in \mathbb{C} \right\} \]

\[
\left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) + \left( \begin{array}{cc} \delta & \gamma \\ -\gamma & \delta \end{array} \right) = \left( \begin{array}{cc} \alpha + \delta & \beta + \gamma \\ -\beta - \delta & \alpha + \gamma \end{array} \right)
\]

real vector subspace of \( M_2(\mathbb{C}) \)

\[
\left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) \cdot \left( \begin{array}{cc} \delta & \gamma \\ -\gamma & \delta \end{array} \right) = \left( \begin{array}{cc} \alpha \delta - \beta \gamma & \alpha \gamma + \beta \delta \\ -\beta \delta - \alpha \gamma & -\alpha \delta + \beta \gamma \end{array} \right)
\]

subalgebra of \( M_2(\mathbb{C}) \),

\[
\det \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) = 1|\alpha|^2 + 1|\beta|^2
\]

\[
\alpha = a + ib \quad \beta = c + id
\]

\[
\det \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) = a^2 + b^2 + c^2 + d^2
\]

\[ T = \left( \begin{array}{cc} \alpha & -\beta \\ \beta & \alpha \end{array} \right) \] in the subalgebra
\[ T T^* = \begin{pmatrix} \alpha \beta \\ -\beta \alpha \end{pmatrix} \begin{pmatrix} \overline{\alpha} & \overline{-\beta} \\ \overline{-\beta} & \overline{\alpha} \end{pmatrix} = \]
\[ = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & 0 \\ 0 & |\alpha|^2 + |\beta|^2 \end{pmatrix} = \det T \cdot I \]

\[ T^{-1} = \frac{1}{\det T} T^* \quad \det T = (\det T)^{\frac{1}{2}} \]

\( A \) is a real division algebra with involution

\[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad i^2 = -1 \]

\[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad j^2 = -1 \]

\[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad k^2 = -1 \]

is a basis of \( A \)

\[ i^* j = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = k \]
\[ i \cdot j = k \quad j \cdot k = i \quad k \cdot i = j \]

\[ -j \cdot k = \hat{k} \]

\[ i \cdot k = -\hat{j} \]

\[ -k \cdot i \]

A is isomorphic to the algebra of quaternions \( \mathbb{H} \).

\[ q = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k = \]

\[ \left( \begin{array}{cc}
  a + ib & c + id \\
  -c + id & a - ib 
\end{array} \right) = \left( \begin{array}{cc}
  \alpha & \beta \\
  -\beta & \alpha 
\end{array} \right) \]

\[ \overline{q} = a \cdot 1 - b \cdot i - c \cdot j - d \cdot k \]

\[ = \left( \begin{array}{cc}
  \frac{\alpha}{|\alpha|^2} & -\frac{\beta}{|\alpha|^2} \\
  \frac{\beta}{|\alpha|^2} & \frac{\alpha}{|\alpha|^2} 
\end{array} \right) \]

involution.
$\mathfrak{m}_n(\mathbb{H})$ algebra of $m \times m$ quaternions

$T^* = \text{transpose}$

$\mathbf{Sp}(n) = \left\{ T \in \mathfrak{m}_n(\mathbb{H}) \mid TT^* = T^*T = 1 \right\}$

Lie subgroup of $GL(m,\mathbb{H})$

$T_1(\mathbf{Sp}(n)) = \left\{ T \mid T \text{ is skew adjoint } \right\}$

$\dim T_1(\mathbf{Sp}(n)) = \frac{m^2-m}{2}$, $4+3m =$

$= 2n^2-2m+3m = 2n^2+m$

$\left( T^* \right)_{jk}^i = \sum_{j'=1}^m T_{j'k}^{j} = \sum_{j=1}^m T_{ij}^{j}T_{jk}^{j}$

$\left( TT^* \right)_{ii} = \sum_{j=1}^m T_{ji}^{j}T_{ji}^{j} = \sum_{j=1}^m |T_{ji}^{j}|^2$
$Sp(m)$ is a closed bounded subset of $M_n(\mathbb{H})$ - compact.

$Sp(m)$ is a compact Lie group - symplectic group.

$Sp(1) - \text{dim}=3$

Quaternions of norm 1,

$$\mathbb{H} = \left\{ \left( \frac{\alpha}{\beta}, \frac{\beta}{\alpha} \right) \mid |\alpha|^2 + |\beta|^2 = 1 \right\}$$

Hence, $Sp(1) = SU(2)$. 