Math 3220-3 Take Home Midterm 1, February 6, 2017

Show all work!

Name:

Problem 1. For any positive integer n, define function

$$f_n(x) = \frac{x}{1 + nx^2}$$

for all real x. Show that the sequence (f_n) converges uniformly to a function f.

Calculate the derivatives of f_n and show that $\lim_{n\to\infty} f_n'(x) = f'(x)$ holds for $x \neq 0$ but fails at $x = 0$.

Problem 2. Let f be a continuous function on $\mathbb{R}$. For each positive integer n, define the function $f_n(x) = f(nx)$ for any real x. Assume that the restrictions of functions f_n to $[0, 1]$ are equicontinuous. What can you say about function f?

Problem 3. Let (f_n) be a sequence of continuous functions on $[a, b]$. Assume that it is uniformly bounded, i.e., there exists $M > 0$ such that $|f_n(x)| \leq M$ for all n and $x \in [a, b]$. Put

$$F_n(x) = \int_a^x f_n(y)\, dy$$

for $x \in [a, b]$. Prove that functions F_n are continuous on $[a, b]$. Show that there is a subsequence of (F_n) which converges uniformly on $[a, b]$.

Problem 4. Let f be a continuous function on $[0, 1]$ such that

$$\int_0^1 f(x)x^n\, dx = 0$$

for all integers $n \geq 0$. Show that $f = 0$!

Problem 5. Let $\mathcal{C}(S^1)$ be the algebra of complex continuous functions on the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane. Consider the subalgebra $\mathcal{A}$ of all functions

$$f(e^{i\phi}) = \sum_{n=0}^{N} c_n e^{in\phi}$$

for real ϕ. Then $\mathcal{A}$ separates points on S^1 and vanishes at no point of S^1. Show that $\mathcal{A}$ is not dense in $\mathcal{C}(S^1)$!

Extra credit: Fill in all details in the sketch of the solution of problem 25. The best solution will be presented in the class!