Math 3220-3 Take Home Midterm 2, March 14, 2021

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Name:

Problem 1. Let $\mathcal{C}(S^1)$ be the algebra of complex continuous functions on the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane. Consider the subalgebra $\mathcal{A}$ of all functions

$$f(e^{i\phi}) = \sum_{n=0}^{N} c_n e^{in\phi}$$

for real ϕ. Then $\mathcal{A}$ separates points on S^1 and vanishes at no point of S^1. Show that $\mathcal{A}$ is not dense in $\mathcal{C}(S^1)$!

Problem 2. Let f be a continuous function on $\mathbb{R}$ periodic with period 2π, given by $f(x) = |x|$ for $-\pi \le x \le \pi$. Using Bessel equality for its Fourier coefficients prove that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

Problem 3. Let A be a linear map from $\mathbb{R}^n$ into $\mathbb{R}$. Show that

(i) there is a unique vector $y \in \mathbb{R}^n$ such that $A(x) = (x \mid y)$ for all $x \in \mathbb{R}^n$;

(ii) $\|A\| = |y|$.

Problem 4. Let f be a function on $\mathbb{R}^2$ defined by

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0); \\ \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

Prove

(i) f is not continuous at $(0,0)$;

(ii) The first partial derivatives of f exist at every point of $\mathbb{R}^2$.

Is f differentiable at $(0,0)$?