Problem 1. Let \( f \) be a continuous function on \( \mathbb{R} \) periodic with period \( 2\pi \), given by \( f(x) = |x| \) for \( -\pi \leq x \leq \pi \). Using Bessel equality for its Fourier coefficients prove that

\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.
\]

Problem 2. Let \( f \) be a periodic continuous function on \( \mathbb{R} \) with period \( 2\pi \). Denote by

\[
\sum_{n \in \mathbb{Z}} c_n e^{inx}
\]

its Fourier series. Show that the following conditions are equivalent:

(i) the function \( f \) is infinitely differentiable;

(ii) for any \( k \geq 0 \) there exists \( M > 0 \) such that \( |c_n||n|^k \leq M \) for all \( n \in \mathbb{Z} \).

Problem 3. Let \( A \) be a linear map from \( \mathbb{R}^n \) into \( \mathbb{R} \). Show that

(i) there is a unique vector \( y \in \mathbb{R}^n \) such that \( A(x) = (x \mid y) \) for all \( x \in \mathbb{R}^n \);

(ii) \( \|A\| = |y| \).

Problem 4. Let \( f \) be a function on \( \mathbb{R}^2 \) defined by

\[
f(x, y) = \begin{cases} 
0 & \text{if } (x, y) = (0, 0); \\
x y & \text{if } (x, y) \neq (0, 0).
\end{cases}
\]

Prove

(i) \( f \) is not continuous at \( 0 \);

(ii) The first partial derivatives of \( f \) exist at every point of \( \mathbb{R}^2 \).

Is \( f \) differentiable at \( (0, 0) \)?