Math 3220-3 Take Home Midterm 1, February 15, 2021

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Name:

Problem 1. Let X be a metric space with metric d. A function $\mathbb{N} \longrightarrow X$ is *a sequence* in X. A point x_0 in X is *a limit* of sequence $\{x_n; n \in \mathbb{N}\}$ if for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $d(x_n, x_0) < \epsilon$.

A sequence is called *convergent* if it has a limit.

Prove that:

(i) A convergent sequence $\{x_n; n \in \mathbb{N}\}$ has only one limit x_0. We put $x_0 = \lim x_n$.

(ii) Let A be a subset of X. Then the closure $\bar{A}$ is the set of all limits of all convergent sequences in A; i.e., $x \in \bar{A}$ if and only if there exists a convergent sequence $\{x_n; n \in \mathbb{N}\}$ such that $x_n \in A$ for any $n \in \mathbb{N}$ and $x = \lim x_n$.

Problem 2. Let f be a continuous map from a topological space X into a topological space Y. Let A be a subset of X. Show that

$$f(\bar{A}) \subset \overline{f(A)}.$$

Show, by example of a function from $\mathbb{R}$ into $\mathbb{R}$, that $f(\bar{A})$ can be a proper subset of $\overline{f(A)}$.

Problem 3. Let X be a hausdorff topological space and $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ a decreasing sequence of compact subsets of X. Let U be an open set in X. If $\bigcap_{n=1}^{\infty} K_n \subset U$, show that there exists n_0 such that $K_n \subset U$ for $n \geq n_0$.

Problem 4. Let f be a continuous real function on $[0, 1]$ such that

$$\int_0^1 f(x) x^n \, dx = 0$$

for all integers $n \geq 0$. Show that $f = 0$!