**Contraction principle**

Let $X$ be a metric space with metric $d$.

A map $\varphi : X \rightarrow X$ is called a contraction if there exists a positive number $c$, $c < 1$, such that

$$d(\varphi(x_1), \varphi(x_2)) \leq c \cdot d(x_1, x_2)$$

for $x_1, x_2 \in X$.

A fixed point of $\varphi$ is $x_0 \in X$ such that $\varphi(x_0) = x_0$.

Remark. A contraction has at most one fixed point.

Assume that $x_1$ and $x_2$ are
fixed points of \( \psi \). Then
\[
d(x_1, x_2) = d(\psi(x_1), \psi(x_2)) \leq c \cdot d(x_1, x_2)
\]
If \( d(x_1, x_2) \neq 0 \implies 1 \leq c \\
what is impossible since \( c < 1 \).
Hence, \( d(x_1, x_2) = 0 \implies x_1 = x_2 \).
The main point of 

contraction principle is

the existence of fixed

points.

Theorem (Contraction principle) Let \( X \) be a
complete metric space. 
Let \( \psi: X \rightarrow X \) be a contraction.
Then $\varphi$ has a (unique) fixed point.

Proof: Let $x_1 \in X$. Construct a sequence $x_n = \varphi^{n-1}(x_1)$. Then

$$d(x_{m+1}, x_m) = d(\varphi(x_m), \varphi(x_{m-1})) \leq c \cdot d(x_m, x_{m-1})$$

for all $m \geq 2$.

By induction, we prove

$$d(x_{m+1}, x_m) \leq c^{m-1}d(x_2, x_1),$$

This is true for $m = 1$.

The induction step:

$$d(x_{m+2}, x_{m+1}) = d(\varphi(x_{m+1}), \varphi(x_m)) \leq c \cdot d(x_{m+1}, x_m) \leq c^m d(x_2, x_1).$$
Let \( n, m \in \mathbb{N} \), \( m < n \).

\[
d(x_{m+1}, x_m) \leq (c^{m-2} + c^{m-3} + \ldots + c^0) \, d(x_2, x_1)
\]

\[
\leq \left( \sum_{k=m-1}^{\infty} c^k \right) \, d(x_2, x_1) =
\]

\[
c^{m-1} \left( \sum_{k=0}^{\infty} c^k \right) \, d(x_2, x_1) =
\]

\[
= \frac{c^{m-1}}{1-c} \, d(x_2, x_1)
\]

Since \( c^{m-1} \to 0 \) as \( m \to \infty \), \( (x_m) \) is a Cauchy sequence in \( X \).

Since \( X \) is complete, \( x_m \to x_0 \).
Since $x_{n+1} = \varphi(x_n)$ for all $n$ and $x_{n+1} \to x_0$ as $n \to \infty$, it follows that $\varphi(x_n) \to x_0$.

Since $d$ is continuous,

$$\lim d(\varphi(x_0), \varphi(x_m)) = d(\varphi(x_0), x_0)$$

On the other hand,

$$0 \leq d(\varphi(x_0), \varphi(x_m)) \leq c \frac{d(x_0, x_m)}{n}$$

as $n \to \infty$

This implies that

$$\lim (\varphi(x_0), \varphi(x_m)) = 0$$

and $d(\varphi(x_0), x_0) = 0 \Rightarrow [\varphi(x_0) = x_0]$
Therefore $x_0$ is a fixed point. We already proved that it is unique.