Higher partial derivatives

Let $f : O \to \mathbb{R}$ be a function defined on an open set $O$ in $\mathbb{R}^m$, $m \geq 2$. Then we can try to calculate higher partial derivatives. For example, we if $i \neq j$ we can calculate

$$\frac{\partial}{\partial x_i} \left( \frac{\partial f}{\partial x_j} \right)$$

and

$$\frac{\partial}{\partial x_j} \left( \frac{\partial f}{\partial x_i} \right).$$

In general, they do not have to be equal!
Example:

\[ f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \end{cases} \]

The numerator and denominator are differentiable functions on \( \mathbb{R}^2 \setminus \{(0,0)\} \). So \( f \) is differentiable there, and therefore continuous.

Moreover, we have

\[ |f(x,y)| = \frac{|x||y|}{x^2+y^2} \cdot |x^2-y^2| \leq 1 \cdot \frac{|x||y|}{x^2+y^2} \cdot (x^2+y^2) = \frac{|x||y|}{x^2+y^2} \rightarrow 0 \]

as \((x,y) \rightarrow (0,0)\).
Hence $f$ is continuous at $(0,0)^3$. It follows that $f$ is continuous on $\mathbb{R}^2$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left( \frac{x y (x^2 - y^2)}{x^2 + y^2} \right) =$$

$$= \frac{\partial}{\partial x} \left( \frac{x^3 y - x y^3}{x^2 + y^2} \right) =$$

$$= \frac{(3 x^2 y - y^3)(x^2 + y^2) - (x^3 y - x y^3) \cdot 2x}{(x^2 + y^2)^2} =$$

$$= \frac{3x^4 y - x^2 y^5 + 3 x^3 y^3 - y^5 - 2x^4 y + 2x^2 y^3}{(x^2 + y^2)^2} =$$

$$= \frac{y(x^4 + 4x^2 y^2 - y^4)}{(x^2 + y^2)^2} \quad \text{for } (x,y) \neq (0,0).$$

Also
\[
\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0)-f(0,0)}{x} = 0
\]

Clearly, \(\frac{\partial f}{\partial x}\) is continuous on \(\mathbb{R}^2 \setminus \{(0,0)\}\).

\[
|\frac{\partial f}{\partial x}(x,y)| = \frac{|y|}{(x^2+y^2)^2} \left|x^4 + 4x^2y^2 - y^4\right| \leq \frac{1|y|}{(x^2+y^2)^2} (x^4 + 4x^2y^2 + y^4) \leq \frac{1|y|}{(x^2+y^2)^2} \cdot 2(x^4 + 3x^2y^2 + y^4) = \frac{2|y|}{(x^2+y^2)^2} (x^4 + y^4)^{\frac{3}{2}} = 2|y|
\]

and

\[
\lim_{(x,y) \to (0,0)} \frac{\partial f}{\partial x}(x,y) = 0.
\]

Hence, \(\frac{\partial f}{\partial x}\) is continuous on \(\mathbb{R}^2\).
\[
\frac{\partial f}{\partial y} (x, y) = \frac{\partial}{\partial y} \left( \frac{x^3 y - x y^3}{x^2 + y^2} \right) = \\
= \frac{(x^3 - 3xy^2)(x^2 + y^2) - 2y(x^3 y - x y^3)}{(x^2 + y^2)^2} \\
= \frac{x^5 - 3x^3 y^2 + x^3 y - 3xy^4 - 2x^3 y^2 + 2xy^4}{(x^2 + y^2)^2} \\
= \frac{x^5 - 4x^3 y^2 - x y^4}{(x^2 + y^2)^2} \\
= x \left( \frac{x^4 - 4x^2 y^2 - y^4}{(x^2 + y^2)^2} \right) \text{ for }(x, y) \neq (0, 0)
\]

Also
\[
\frac{\partial f}{\partial y} (0, 0) = \lim_{y \to 0} \frac{0}{0} = 0
\]

Hence \( \frac{\partial f}{\partial y} \) exists on \( \mathbb{R}^2 \).
Clearly, \( \frac{\partial f}{\partial y} \) is continuous on \( \mathbb{R}^2 \setminus \{(0,0)\} \).

We also have

\[
\left| \frac{\partial f}{\partial y} (x,y) \right| = \frac{|x|}{(x^2+y^2)^2} x^4-4x^2y^2-y^4 \leq \\
\leq \frac{|x|}{(x^2+y^2)^2} (x^4+4x^2y^2+y^4) = \\
\leq \frac{2|x|}{(x^2+y^2)^2} (x^4+2x^2y^2+y^4) = \\
= \frac{2|x|}{(x^2+y^2)^2} \cdot (x^2+y^2)^2 = 2|x| \\
\]

so \( \lim_{(x,y) \to (0,0)} \frac{\partial f}{\partial y} = 0 \).

Hence \( \frac{\partial f}{\partial y} \) is also continuous on \( \mathbb{R}^2 \).
Now
\[ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)(x, y) \] and \[ \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)(x, y) \]
eXist for \((x, y) \neq 0\).

Moreover for \(y \neq 0\)
\[ \frac{\partial f}{\partial x}(0, y) = -\frac{y^3}{y} = -y \]

Hence
\[ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)(0, 0) = -1. \]

Also, for \(x \neq 0\)
\[ \frac{\partial f}{\partial y}(x, 0) = \frac{x^5}{x^4} = x \]

Hence
\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)(0,0) = \lim_{x \to 0} \frac{x - 0}{x} = 1.
\]

Hence
\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)(0,0) = -\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)(0,0).
\]

By direct calculation also we can check that these two derivatives exist and are equal for \((x, y) \neq (0, 0)\).

The last statement is also consequence of next theorem, since \(\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)\) and \(\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)\) are continuous on \(\mathbb{R}^2 - (0, 0)\).
Theorem. Let \( O \) be an open set in \( \mathbb{R}^2 \). Assume that \( \frac{\partial f}{\partial x} > \frac{\partial f}{\partial y} \) and \( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \) exist for any \((x,y) \in O\). Assume that \( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \) is continuous at some \((a,b) \in O\). Then \( \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \) exists at \((a,b)\) and

\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)(a,b) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)(a,b).
\]

Remark. If \( f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > 0 \), \( \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \) and \( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \) exist and are continuous on \( O \),
we have
\[ \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right). \]

Hence, we put
\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \]
where the order of $x$ and $y$ is irrelevant.

**Proof of the theorem**

Consider the rectangle $Q$ with vertices $(a,b)$, $(a+h,b)$, $(a,b+k)$ and $(a+h,b+k)$.

Put $m(t) = f(t,b+k) - f(t,b)$. 
Since $\frac{\partial f}{\partial x}$ exists, $u$ is differentiable for $a \leq t \leq a+h$. By mean value theorem, there exists $x \in [a, a+h]$ such that

$$m(a+h) - m(a) = m'(x) \cdot h = \left(\frac{\partial f}{\partial x}(x,b+k) - \frac{\partial f}{\partial x}(x,b)\right) \cdot h.$$ 

Applying mean value theorem again

$$\frac{\partial f}{\partial x}(x,b+k) - \frac{\partial f}{\partial x}(x,b) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x}\right)(x,y) \cdot k$$

for some $y \in [\theta, b+k]$. It follows that

$$m(a+h) - u(a) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x}\right)(x,y) \cdot h \cdot k.$$
Therefore, we have

\[ f(a+h,b+k) - f(a,b+k) - f(a+h,b) + f(a,b) = \]

\[ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)(x,y).h.k \]

for some \((x,y) \in \mathbb{Q}\).

**Proposition**

\[ \Delta(f, \mathbb{Q}) = f(a+h,b+k) - f(a,b+k) - f(a+h,b) + f(a,b) \]

Then

\[ \frac{\Delta(f, \mathbb{Q})}{h.k} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)(x,y). \]

Since \( \frac{\partial}{\partial y} (\frac{\partial f}{\partial x}) \) is continuous at \((a,b)\), we have
\[
\lim_{(h,k) \to (0,0)} \frac{\Delta(f,Q)}{h \cdot k} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)_{(a,b)}
\]

Let \( \varepsilon > 0 \). Then there exist \( \delta \) such that
\[
\sqrt{h^2 + k^2} < \delta \implies \left| \frac{\Delta(f,Q)}{h \cdot k} - A \right| < \varepsilon.
\]

Fix \( h \). Let \( k \to 0 \). Then
\[
\lim_{k \to 0} \frac{\Delta(f,Q)}{k} = \lim_{k \to 0} f(a+h,b+k) - f(a+h,b) - f(k+h,b) = 0,
\]
\[
- \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}
\]
\[
= \frac{\partial f}{\partial y} (a+h,b) - \frac{\partial f}{\partial y} (a,b).
\]

Hence, we have
Therefore

\[ \frac{e}{\partial x} \frac{\partial f}{\partial y}(a, e) = A \]

as \( h \to 0 \).

Therefore, if \( O \subset \mathbb{R}^n \) is open and \( f : O \to \mathbb{R} \).

If \( \exists f, 1 \leq i \leq n, \exists \exists f, 1 \leq i, j \leq n \), exist and are continuous we have

\[ e_i \partial f = e_j \partial_i f \]

for all \( 1 \leq i, j \leq n \).