\[ \mathbb{R}^m \times \mathbb{R} = \mathbb{R}^{m+1} \]

\[ \uparrow \quad \text{natural topology} \]

\[ \text{product topology} \]

\[ \| (x_1, \ldots, x_m, x_{m+1}) \| = \]

\[ \| (x_1, \ldots, x_m, 0) + (0, \ldots, 0, x_{m+1}) \| \leq \]

\[ \| (x_1, \ldots, x_m, 0) \| + \| (0, \ldots, 0, x_{m+1}) \| = \]

\[ = \| (x_1, \ldots, x_m) \| + |x_{m+1}|. \]

\[ d((x_1, \ldots, x_{m+1}), (y_1, \ldots, y_{m+1})) = \]

\[ = d((x_1, \ldots, x_m), (y_1, \ldots, y_m)) + |x_{m+1} - y_{m+1}| \]

\[ \Rightarrow \quad \text{ball of radius } 2 \text{ centered at } (x_1, \ldots, x_{m+1}) \text{ contains the} \]

\[ \text{ball of radius } \varepsilon/2 \text{ centered at } (x_1, \ldots, x_m) \times (x_{m+1} - \varepsilon/2, x_{m+1} + \varepsilon/2) \]
⇒ Open set in natural topology is open in the product topology.

\[
\| (x_1, \ldots, x_m, 0) \| = \sqrt{x_1^2 + \ldots + x_m^2} \leq \sqrt{x_1^2 + \ldots + x_m^2} = \| (x_1, \ldots, x_m) \| \\
\| (0, \ldots, 0, x_{m+1}) \| = \sqrt{x_{m+1}^2} \leq \| (x_1, \ldots, x_{m+1}) \|
\]

⇒ ball of radius \( \varepsilon \) centered at \((x_1, \ldots, x_m)\) is contained in the product of the ball of radius \( \varepsilon \) centered at \((x_1, \ldots, x_m)\) and \((x_{m+1} - \varepsilon, x_{m+1} + \varepsilon)\).
This implies that an open set in product topology is open in the natural topology. Hence, the natural topology of $\mathbb{R}^{n+1}$ is the product topology of $\mathbb{R}^n \times \mathbb{R}$.

**Theorem** $[a_1, b_1] \times \ldots \times [a_m, b_m] \subset \mathbb{R}^n$ is compact.

**Proof**: We proved this for $n=1$. 
General case follows by induction. \( \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \) as topological spaces. By induction assumption, \([a_1, b_1] \times \ldots \times [a_{n-1}, b_{n-1}]\) is compact. Hence, 
\([a_1, b_1] \times \ldots \times [a_{n-1}, b_{n-1}] \times [a_n, b_n]\) is compact, by the theorem we proved last time.

A set \( S \) in \( \mathbb{R}^n \) is bounded if it is contained in some ball centered at \( 0 \) (this depends on metric, not on topology!).
**Theorem (Heine-Borel)**

Let \( S \subset \mathbb{R}^n \). The following properties are equivalent:

(i) \( S \) is compact;

(ii) \( S \) is closed and bounded.

**Proof:** \( \mathbb{R}^n \) is Hausdorff. Hence, if it is compact, it is closed.

\[
S = \bigcup_{m=1}^{\infty} B(0, m).
\]

By compactness, \( S \subset B(0, N) \) for some \( N > 0 \), i.e., \( S \) is bounded.

If \( S \) is closed and bounded, \( S \) is a closed subset of...
a sufficiently large box \([a_1, b_1] \times \cdots \times [a_n, b_n]\). Since this box is compact, \(S\) is compact.
Weierstrass theorem

Can define

\[ \alpha : [0,1] \rightarrow [a,b] \]

\[ \alpha (t) = a + (b-a)t \]

\[ \alpha^{-1}(s) = \frac{s-a}{b-a} \text{ - inverse map.} \]

Theorem: Let \( f \) be a continuous function on \([a,b]\). Then there exist function on \([a,b]\). Then for any \( \epsilon > 0 \) there exists a polynomial \( P \) such that \( \|f-P\| < \epsilon \) in \( C([a,b]) \).

(Therefore, polynomials are dense in \( C([a,b]) \).

Proof: I can assume that \( a=0, b=1 \).
Can assume that $f(0) = f(1) = 0$.

**Proof:** Put $g(x) = f(0) + (f(1) - f(0))x$ linear.

$F(x) = f(x) - g(x)$, $F(0) = F(1) = 0$.

$F(0) = f(0) - g(0) = 0$, $F(1) = f(1) - g(1) = 0$.

Can extend $f$ to be 0 outside $[0, 1]$.

Then $f$ is continuous on $\mathbb{R}$.

![Graph of f with Q_m(x) = c_m(1-x^2)^m](image)

$Q_m(x) = c_m(1-x^2)^m$ is positive on $[-1, 1]$. Can pick $c_m > 0$ such that

$$\int_{-1}^{1} Q_m(x) dx = 1.$$ 

$$P_m(x) = \int_{-1}^{x} f(x+t) Q_m(t) dt$$

for $0 \leq x \leq 1$.

$f$ is zero outside $[0, 1]$.
\[ t \mapsto f(x+t) \] is 0 for \( t \leq -x, t \geq 1-x. \]

Hence

\[ P_m(x) = \int_{-x}^{1-x} f(x+t) Q_m(t) \, dt. \]

Make change of variables \( s = t+x \)
\[ ds = dt \]

\[ P_m(x) = \int_0^{1-2x} f(s) Q_m(s-x) \, ds, \]

\[ Q_m(s-x) = C_m \left( 1 - (s-x)^2 \right)^n \]

is polynomial in \( s \) and \( x \)

\[ Q_m(s-x) = \sum a_{pq} s^p x^q \]

\[ \Rightarrow P_m(x) = \sum a_{pq} \left( \int_0^1 f(s) s^p \, ds \right) x^q \]

is a polynomial in \( x \).

\[ \left| P_m(x) - f(x) \right| = \left| \int_0^1 f(x+t) Q_m(t) \, dt - \right| \]

\[ - f(x) \int_{-1}^1 Q_m(t) \, dt = \]
\[
\frac{1}{L} \left| \int_{-\frac{1}{L}}^{1} (f(x+t) - f(x)) \Omega_m(t) \, dt \right| \\
\leq \int_{-1}^{1} |f(x+t) - f(x)| \Omega_m(t) \, dt, \\
\text{since } f \text{ is uniformly continuous on } \mathbb{R}, \text{ there exists } \delta > 0 \text{ such that } \\
|t| < \delta \text{ implies } |f(x+t) - f(x)| < \frac{\varepsilon}{2}, \\
P_m(x) - f(x) \leq \int_{-\delta}^{\delta} |f(x+t) - f(x)| \Omega_m(t) \, dt + \\
\int_{-\delta}^{\delta} |f(x+t) - f(x)| \Omega_m(t) \, dt + \\
\int_{-1}^{1} |f(x+t) - f(x)| \Omega_m(t) \, dt < \frac{\varepsilon}{2}. \\
f \text{ is bounded, } \exists M > 0 \text{ such that }
\[
|P_m(x) - f(x)| \leq M \int_{-1}^{1} Q_m(t) \, dt + \varepsilon/2 + M \int_{-\delta}^{\delta} Q_m(t) \, dt \\
\leq \varepsilon/2 + 2M \int_{-\delta}^{\delta} \frac{1}{\delta} \, dt \\
(\text{since } Q_m \text{ is even}).
\]

If we prove that \( \int_{-\delta}^{\delta} Q_m(t) \, dt \to 0 \) as \( m \to \infty \), there exists \( m \) such that \( 2M \int_{-\delta}^{\delta} Q_m(t) \, dt < \varepsilon/2 \) for \( m \geq m_0 \).

Hence, for \( m \geq m_0 \), we have
\[
|P_m(x) - f(x)| < \varepsilon \text{ for all } x \in [-1, 1].
\]
This implies that \( \|P_m - f\| < \varepsilon \) for \( m \geq m_0 \).

It remains to prove \( \ast \)