Dimension of a manifold

Connectedness - Let $X$ be a topological space. $X$ is connected if $X$ is not equal to $U \cup V$ where $U, V$ are nonempty open, disjoint sets.

Example: Open ball in $\mathbb{R}^n$ is connected.

Lemma. Let $x \in X$. Denote by $U$ the family of all connected subsets of $X$ containing $x$.
Then the union of all sets in $U$ is connected.

Proof. Assume that $Y$ is that
union, \( Y = U \cap V \), \( U \cup V \neq \emptyset \)
and \( U \cap V = \emptyset \) \( U \cup V \) open in \( Y \).

Let \( Z \in U \). Then \( Z \cap U \), \( Z \cap V \)
are open in \( Z \), \( (Z \cap U) \cap (Z \cap V) = \emptyset \)
and \( (Z \cap U) \cup (Z \cap V) = Z \).

Since \( Z \) is connected, this is possible only if \( Z \cap U = \emptyset \) or
\( Z \cap V = \emptyset \).

Assume that \( x \in U \). Then \( x \in Z \cap U \)
\( \Rightarrow Z \cap V = \emptyset \Rightarrow Z \subset U \).

Hence, union of all \( Z \) is in \( U \).
\( \Rightarrow V = \emptyset \). Therefore, \( Y \) is
connected. \( \Box \)
The set $Y$ is the connected component of $x$.

Let $y \in Y$. Then the connected component $W$ of $y$ contains $Y$ (since it is connected).

It follows that $y \in Y \subset W$.

Therefore, $W \subseteq U$ and $W \subseteq Y$.

Hence, $W = Y$.

⇒ $Y$ is connected component of each of its points.

$X$ is a disjoint union of all of its connected components.

**Theorem.** Connected components of a differentiable manifold
are open (and closed).

Proof. Let $M$ be a manifold and $N$ a connected component of $M$. Let $x \in N$. Then there exists a chart $c = (U, \varphi, \pi)$ around $x$ such that $\varphi(U)$ is a ball. $\implies \varphi(U)$ is connected $\implies U$ is connected. $\implies U \subset N$, $\implies N$ is open. All connected components are open. $X \setminus N$ is a union of components. $\implies N$ is also closed. $\square$
M manifold, \( x \in M \)
\( c = (U, \varphi, n) \) chart around \( x \).

If \( d = (V, \psi, m) \) is another chart around \( x \)
\( \varphi \circ \psi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V) \)
is a diffeomorphism \( \Rightarrow n = m \)
\( \Rightarrow \dim_x M = n \)-dimension of \( M \) at \( x \).

\( \circ \ x \rightarrow \dim_x M \) is a locally constant function (it is constant on a neighborhood of \( x \)).

Assume that \( \dim_x M \) has two different values \( n, m \).
Then \( U = \{ x \in M \mid \dim_x M = n \} \)
\( V = \{ x \in M \mid \dim_x M \neq n \} \)
are open sets in \( M \), \( U \cup V = M \)
\( U, V \neq \emptyset \) and \( U \cap V = \emptyset \).
Hence \( M \) is not connected.

\[ \Rightarrow \quad \text{Local dimension} \ \dim_x M \]
is constant on connected components of a manifold.

If the manifold is connected, \( \dim_x M = \dim M \)
dimension of \( M \).
Products.

- $M, N$ are manifolds
- $M \times N$ - product
- Topological space
- $U \subseteq M \times N$ is open if for any $(x, y) \in U$, there exist open $U_x \ni x$ in $M$, $V_y \ni y$ in $N$ such that $U_x \times V_y \subseteq U$.
- Define charts on $M \times N$

$c = (U, \varphi, m)$
$d = (V, \psi, n)$

$c \times d = (U \times V, \varphi \times \psi, m+n)$

This defines on $M \times N$ a structure of differentiable manifold. - Product manifold
of $M$ and $N$, 

Lie groups

A Lie group $G$ is a

(a) group;

(b) manifold;

$m : M \times M \rightarrow M$

is differentiable map

$i : M \rightarrow M$ \hspace{1cm} i(a) = a^{-1}

is a differentiable map.