Second reduction

\[ U \rightarrow \emptyset \rightarrow V \]

Assume that for any \( a \in V \) there exists an open neighborhood \( V_a \) of \( a \), \( V_a \subset V \) such that the formula holds for functions \( f \in C_0(\mathbb{R}^n) \) such that
Let \( f \in C_c(\mathbb{R}^n) \) with 
\[
\text{supp } f \subset V_a.
\]

Then \( (V_a; \alpha \in K) \) is an open cover of \( K \). Let 
\[
(\psi_1, \ldots, \psi_m)
\]
be a partition of unity subordinated to the cover \( (V_a; \alpha \in K) \).

Then
\[
\sum_{i=1}^{m} \psi_i(y) = 1 \text{ for } y \in K.
\]

Moreover,
\[
f(y) = \sum_{i=1}^{m} \psi_i(y) f(y) = \sum_{i=1}^{m} (\psi_i f)(y), \quad y \in \mathbb{R}^n.
\]
Put $f_i = f \cdot q_i$. Then $\text{supp } f_i \subset V_i$ for some $a_i \in K$.

By the assumption, we have

$$\int_{\mathbb{R}^m} f_i(x) |J(x)| \, dx = \int_{\mathbb{R}^m} f_i(x) \, dx = \int_{\mathbb{R}^m} f_i(x) \, dx$$

for $1 \leq i \leq m$. Hence

$$\int_{\mathbb{R}^m} f(x) \, dx = \int_{\mathbb{R}^m} \sum_{i=1}^{m} f_i(x) \, dx = \sum_{i=1}^{m} \int_{\mathbb{R}^m} f_i(x) \, dx =$$

$$= \sum_{i=1}^{m} \int_{\mathbb{R}^m} f_i(x) \, dx = \sum_{i=1}^{m} \int_{\mathbb{R}^m} f_i(x) |J_{\tau}(x)| \, dx =$$
\[
= \int \sum_{i=1}^{\infty} f_i(T(x)) \left| \mathcal{J}_T(x) \right| \, dx = \\
= \int f(T(x)) \left| \mathcal{J}_T(x) \right| \, dx \\
\text{So, the formula holds for } f.
\]

(This is a typical example of reducing a global statement to local using partition of unity.)
**Primitive maps**

Let \( \mathcal{G} : \mathcal{U} \rightarrow \mathcal{V} \), \( \mathcal{U} \) and \( \mathcal{V} \) open in \( \mathbb{R}^m \). Then

\[
\begin{array}{c}
\text{U} \\
\downarrow \mathcal{G} \\
\text{V}
\end{array}
\]

\( \mathcal{G} (x) = (G_1 (x), \ldots, G_m (x)) \)

where

\( G_i : \mathbb{R}^m \rightarrow \mathbb{R} \), \( 1 \leq i \leq m \).

We say that \( \mathcal{G} \) is **primitive** if there exists \( m \), \( 1 \leq m \leq m \), such that

\( G_i (x_1, \ldots, x_m) = x_i \)

for \( 1 \leq i \leq m \), \( i \neq m \).

In this case

\( G_m (x) = g (x) \)

is a function on \( \mathcal{U} \).

Assume, in addition, that
$G$ is a continuously differentiable bijection of $U$ onto $V$. Then

$$G'(x) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \ddots & \ddots \\
0 & \cdots & 1 \\
0 & \cdots & 0
\end{bmatrix}$$

Hence, we have

$$J_G(x) = \det G'(x) = \partial_{x} g(x).$$

Assume that $J_G(x) \neq 0$ on $U$. Then we have

$$\partial_{x} g(x) \neq 0$$

on $U$, let $f$ be a continuous
function with compact support in $V$. Then $f \circ G$ is a continuous function with compact support in $V$. Moreover, we have

$$\int f(G(x)) |J_G(x)| \, dx = \int \ldots \int f(G(x)) \left( \frac{\partial m g(x)}{\partial x_m} \right) \, dx \ldots \, dx$$

since the result doesn't depend on the order of integration.

Fix $x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n$. Then $g(x_{m}) = g(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n)$ is a differentiable function on an open set in $IR^n$. Then

$$g'(x_m) = \left( \frac{\partial m g}{\partial x_m} \right)(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n)$$
Put
\[ \varphi(x_m) = f(x_1, \ldots, x_m, \ldots, x_m) \].
Then
\[ \varphi(\gamma(x_m)) = f(x_1, \ldots, g(x), \ldots, x_m) \]
Hence, by 1-dim. version of change of variables formula, we know that
\[ \int f(x_1, \ldots, x_{m-1}, g(x), \ldots, x_m) \left| \frac{\Delta g(x_1, \ldots, x_m, x_{m-1})}{dx_m} \right| dx_m = \]
\[ = \int \varphi(\gamma(x_m)) \left| \frac{\Delta \gamma(x)}{dx_m} \right| dx_m = \]
\[ = \int \varphi(x_m) dx_m = \]
\[ = \int f(x_1, \ldots, x_m, \ldots, x_m) dx_m \]
Plugging this in the previous expression we get
\[ \int f(G(x)) |J_G(x)| \, dx = \int f(x) \, dx. \]

Hence, the change of variables formula holds for primitive maps.