Let \( f(x_1, \ldots, x_m) = \prod_{i=1}^{m} f_i(x_i) \) where \( f_i : [a_i, b_i] \to \mathbb{R} \) are continuous functions. Then \( f \) is continuous on \( I^n \).

Moreover, we have

\[
\sum_{a_i} \int_{a_i}^{b_i} f(x_1, \ldots, x_m) \, dx_i = \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} f(x_1, \ldots, x_m) \, dx_1 \cdots dx_m = \prod_{i=1}^{m} \int_{a_i}^{b_i} f_i(x_i) \, dx_i
\]

This implies that

\[
\sum_{a_i} \int_{a_i}^{b_i} f(x_1, \ldots, x_m) \, dx_i = \prod_{i=1}^{m} \int_{a_i}^{b_i} f_i(x_i) \, dx_i
\]
Hence, for any two permutations \( \pi, \pi' \) we have

\[
I_{\pi}(f) = I_{\pi'}(f)
\]

for all special functions \( f \).

Let \( A \) be the subalgebra of \( C(I) \) spanned by functions

\[
f(x_1, \ldots, x_m) = f_1(x_1) \cdots f_m(x_m),
\]

\( f_i \in C([a_i, b_i]) \), \( 1 \leq i \leq m \).

Then \( f \) contains constants.

Moreover, if \( (x_1, \ldots, x_m) \) and \( (y_1, \ldots, y_m) \) are two different points in \( I^m \), there exists \( 1 \leq i \leq m \), such that \( x_i \neq y_i \).
Let \( \varphi : [a_i, b_i] \rightarrow \mathbb{R} \)

\[ \varphi(x_i) \neq \varphi(y_i). \]

Then \( f(x_1, \ldots, x_m) = \varphi(x_i) \) in \( A \) and

\[ f(x_1, \ldots, x_m) \neq f(y_1, \ldots, y_m). \]

Hence, \( A \) differs points in \( A \).

By Stone-Weierstrass theorem \( A \) is dense in \( C(I^n) \).

Put

\[ \text{vol}(I^n) = |b_1 - a_1| \ldots |b_m - a_m|. \]

Then we have
$$\left| \mathbf{I}_\pi(f) \right| \leq \sum_{\sigma(\pi)} \left( \sum_{\sigma(\pi)} |f(x_1, \ldots, x_n)| \, dx_{\sigma(1)} \right) \ldots dx_{\sigma(n)} \leq \|f\| \cdot \text{vol}(I^n).$$

Let $f \in C(I^n)$, and $\varepsilon > 0$. Then there exists $g \in A$ such that $\|f - g\| < \varepsilon$.

This implies that
$$\left| \mathbf{I}_\pi(f) - \mathbf{I}_\pi(g) \right| = \left| \mathbf{I}_\pi(f - g) \right| \leq \|f - g\| \cdot \text{vol}(I^n) < \varepsilon \cdot \text{vol}(I^n)$$

for any permutation $\pi$.

Let $\pi$ and $\pi'$ be two permutations. Then
\[ |I_\pi(f) - I_\pi(g)| \leq \varepsilon \cdot \text{vol}(I^n) \]

and
\[ |I_\pi'(f) - I_\pi'(g)| \leq \varepsilon \cdot \text{vol}(I^n). \]

Since \( g \in A \)
\[ I_\pi(g) = I_\pi'(g). \]

Hence, we have
\[ |I_\pi(f) - I_\pi'(f)| = |I_\pi(f) - I_\pi(g) + I_\pi(g) - I_\pi'(f)| \leq |I_\pi(f) - I_\pi(g)| + |I_\pi'(f) - I_\pi'(g)| \leq 2 \varepsilon \cdot \text{vol}(I^n). \]

It follows that
\[ I_\pi(f) = I_\pi'(f) \]
for any \( f \in C(I^n) \).
Therefore, linear forms $I_\pi$ are all equal on $C(I^n)$. We define

$$I(f) = I_\pi(f)$$

for any permutation $\pi$. Hence $I$ is an iterated integral and it doesn’t depend on order of integration!