Theorem. Let $O$ be an open set in $\mathbb{R}^n$ and $F : O \to \mathbb{R}^n$ a continuously differentiable function. Assume that $F'(a)$, $a \in O$, is invertible. Then there exist open neighborhoods $U$ of $a$ and $V$ of $b = F(a)$ such that

(i) $F : U \to V$ is a bijection,

(ii) the inverse function $G : V \to U$ is continuously differentiable at $b$ such that

$$G'(b) = F'(a)^{-1}.$$
First we need some properties of operator norm.

\[ A e_j = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} = \begin{bmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{bmatrix} \]

\[ \|A e_j\|^2 \leq \|A\|^2 \|e_j\|^2 = \|A\|^2 \]

\[ \sum_{i=1}^{\infty} |A_{ij}|^2 \leq \|A\|^2 \]

\[ \Rightarrow |A_{ij}|^2 \leq \|A\|^2 \Rightarrow |A_{ij}| \leq \|A\| \]

\[ M = \max_{i,j} |A_{ij}| \leq \|A\| \]
\[ \|Ax\|^2 = \sum_{i=1}^{m} (Ax)_{i}^2 = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} A_{ij} x_{j} \right)^2 \]

\[ \leq \sum_{i=1}^{m} \left( \sum_{j=1}^{n} |A_{ij}| |x_{j}| \right)^2 \leq M^2 \sum_{i=1}^{m} \left( \sum_{j=1}^{n} |x_{j}| \right)^2 \]

\[ \leq M^2 \sum_{i=1}^{m} n \cdot \sum_{j=1}^{n} |x_{j}|^2 = n^2 M^2 \|x\|^2 \]

by Cauchy-Schwarz

\[ \Rightarrow \quad \|Ax\|^2 \leq n^2 M^2 \|x\|^2 \]

\[ \Rightarrow \quad \|Ax\| \leq n \cdot M \cdot \|x\| \]

\[ \Rightarrow \quad \|Ax\| \leq n \cdot M \]

\[ \boxed{M \leq \|Ax\| \leq n \cdot M} \]

If \( x \mapsto A(x) \) is continuous, for \( \varepsilon > 0 \), \( \exists \delta > 0 \), \( |A(x)_{ij} - A(x_0)_{ij}| < \varepsilon \) as \( \|x - x_0\| < \delta \) \( \Rightarrow \|A(x) - A(x_0)\| < m \cdot \varepsilon \)

\( x \mapsto \|B(x)\| \) is continuous.
Proof. \( A = F'(a) \).

Consider, for \( y \in \mathbb{R}^m \), the function

\[
\varphi(x) = x + A^{-1}(y - F(x))
\]

\[
\varphi(x) = x \iff A^{-1}(y - F(x)) = 0 \iff y - F(x) = 0 \iff y = F(x).
\]

\[
\varphi'(x) = I - A^{-1}F'(x) =
\]

\[
= A^{-1}(A - F'(x)),
\]

\[
\|\varphi'(x)\| \leq \|A^{-1}\| \cdot \|A - F'(x)\|.
\]

Since \( F'(x) \) is continuous on \( \mathbb{R} \),
there exists an open ball \( U \) centered at \( a \)

such that \( \|A - F'(x)\| \leq \frac{1}{2\|A^{-1}\|} \),

\[
\Rightarrow \|\varphi'(x)\| \leq \|A^{-1}\| \cdot \frac{1}{2\|A^{-1}\|}.
\]
Since $U$ is convex, we have
\[ \| \varphi(c) - \varphi(d) \| \leq \frac{1}{2} \| c - d \|, \]
for $c, d \in U$. Hence, $\varphi$ is a contraction. Hence, it can have at most one fixed point in $U$.

So, for $y \in \mathbb{R}^m$, the equation $y = F(x)$ has at most one solution in $U$.

\[ \Rightarrow F: U \rightarrow \mathbb{R} \text{ is 1-to-1 (i.e., an injection)}, \]