Theorem. Equivalent

(i) $F$ is continuously differentiable on $U$

(ii) $\partial_i F_j$, $1 \leq i \leq n$, $1 \leq j \leq m$, exist and are continuous on $U$.

Proof. We proved $(i) \Rightarrow (ii)$.

Converse $(ii) \Rightarrow (i)$:

Assume that $\partial_i F_j$ exist and are continuous on $U$.

1. $F = (F_1, \ldots, F_m)$. It is enough to show that $F_i$ are continuously differentiable

\[
\|F(x_0 + h) - F(x_0) - Ah\| \leq \sum_{i=1}^{n} \left| F_i(x_0 + h) - F_i(x_0) - \sum_{j=1}^{m} A_{ij} h_j \right|^2
\]
\[
\frac{\|F(x_0 + h) - F(x_0) - \mathbf{Ah}\|^2}{\|h\|^2} = \sum_{i=1}^{m} \frac{|F_i(x_0 + h) - F_i(x_0) - \sum A_{ij}h_j|^2}{\|h\|^2} \\
\text{Since } F_j \text{ are differentiable at } x_0, \\
\Rightarrow F \text{ is differentiable at } x_0 \\
F'(x_0) = \left( \partial_j F_j(x_0) \right) \Rightarrow F' \text{ is continuous}
\]

Can assume that \( m = 1 \).

\[
F(x_0 + h) - F(x_0) = \sum_{j=1}^{m} F(x_0 + \theta_j) - F(x_0 + \theta_j - 1) \\
h = (h_1, \ldots, h_m) \\
\nu = \sum h_i e_i, \quad \nu_m = h, \quad \nu_j = (h_j, 0, \ldots, 0)
\]
\[ F(x_0 + h) - F(x_0) = \sum_{k=1}^{m} \left( F(x_0 + \nu_k - \nu_{k-1}) \right) = \sum_{k=1}^{m} h_k \partial_k F(x_0 + \nu_k - \nu_{k-1} + \Theta_k h_k e_k) \quad \text{for some } 0 \leq \Theta_k \leq 1 \]

- By mean value theorem.

Can assume that

\[ |\partial_0 F(x_0 + h) - \partial_0 F(x_0)| < \frac{\varepsilon}{m} \]

\[ |F(x_0 + h) - F(x_0) - \sum_{k=1}^{m} \partial_k F(x_0) h_k| \leq \sum_{k=1}^{m} \left| \partial_k F(x_0 + \nu_{k-1} + \Theta_k h_k e_k) - \partial_k F(x_0) h_k \right| \leq \sum_{k=1}^{m} \frac{\varepsilon}{m} |h_k| = \frac{\varepsilon}{m} \sum_{k=1}^{m} |h_k| \leq \frac{\varepsilon}{m} \cdot \sqrt{m} \cdot \|h\| = \frac{\varepsilon}{m} \cdot \|h\| \]

By Cauchy-Schwarz inequality

\[ F \text{ is differentiable!} \]
Inverse function theorem

\[ F : \mathbb{R}^m \rightarrow \mathbb{R}^m \quad G : \mathbb{R}^m \rightarrow \mathbb{R}^m \]

\[ \text{and } G = F^{-1}, \text{ i.e.} \]

\[ (G \circ F)(x) = x \]

\[ \Rightarrow \text{ By the chain rule} \]

\[ G'(F(x_0)) \circ F'(x_0) = I \]

If \( y = F(x) \), \( (F \circ G)(y) = y \)

\[ \Rightarrow F'(G(y_0)) \circ G'(y_0) = I \]

\( F'(x_0) \) is an invertible linear map and \( m = m \).

\[ F'(x_0)^{-1} = G'(F(x_0)) \]
Theorem. Let $O$ be an open set in $\mathbb{R}^n$ and $F: O \to \mathbb{R}^n$ a continuously differentiable function. Assume that $F'(a), a \in O$, is invertible. Then there exist open neighborhoods $U$ of $a$ and $V$ of $b = F(a)$ such that

(i) $F: U \to V$ is a bijection;

(ii) the inverse function $G: V \to U$ is continuously differentiable at

$$G(b) = F(a)^{-1}.$$