Math 2270-3 Take Home Midterm 1, September 30, 2018

Show all work!

Name:

Problem 1. Let $M_2(\mathbb{R})$ be the set of all real 2×2 matrices with usual addition and multiplication.

(a) Show that $M_2(\mathbb{R})$ is a ring with zero equal to

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and identity

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Show that the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$.

(c) Show that $M_2(\mathbb{R})$ is not a field.

Problem 2. Consider the space $M_{nm}(\mathbb{R})$ of all real matrices with n rows and m columns. It is a real vector space with standard addition and multiplication.

(a) Let E_{ij} be a matrix in $M_{nm}(\mathbb{R})$ with all matrix coefficients equal to 0 except the (i, j) coefficient which is equal to 1. Show that

$$\{E_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$$

is a basis of the vector space $M_{nm}(\mathbb{R})$.

(b) What is the dimension of the vector space $M_{nm}(\mathbb{R})$?

Problem 3. Let P be a linear map from vector space V into itself. Assume that $P^2 = P$. Show that V is the direct sum of $\ker P$ and $\operatorname{im} P$.

Problem 4. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}.$$

Find:

(a) A basis of the kernel of this matrix;

(b) the dimension of the kernel;

(c) a basis of the the image;

(d) the dimension of the image.

Problem 5. Let V be a finite dimensional vector space over $\mathbb{R}$ with positive definite scalar product $(\cdot \mid \cdot)$. Let $\|\cdot\|$ be the corresponding norm. Prove that:

(a) the norm satisfies the *parallelogram law*:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

for any two vectors $u, v \in V$;

(b) The scalar product satisfies

$$(u|v) = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2)$$

i.e., it is completely determined by the norm.