ON THE COHOMOLOGICAL DIMENSION OF THE LOCALIZATION FUNCTOR

Henryk Hecht, Dragan Miličić

Department of Mathematics, University of Utah

Abstract. The left cohomological dimension of the localization functor is infinite for singular infinitesimal characters.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and X the flag variety of $\mathfrak{g}$, i. e. the variety of all Borel subalgebras in $\mathfrak{g}$. Let $\mathfrak{h}$ be the (abstract) Cartan algebra of $\mathfrak{g}$, Σ the root system in $\mathfrak{h}^*$ and Σ^+ the set of positive roots determined by the condition that the homogeneous line bundles $\mathcal{O}(-\mu)$ on X corresponding to dominant weights μ are positive. Denote by W the Weyl group of Σ. By a well-known result of Harish-Chandra the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ is isomorphic to the Weyl group invariants $I(\mathfrak{h})$ in the symmetric algebra $S(\mathfrak{h})$. Therefore, the space of all maximal ideals in $\mathcal{Z}(\mathfrak{g})$ can be identified with the W-orbits in $\mathfrak{h}^*$. Let θ be such an orbit in $\mathfrak{h}^*$, and denote by J_θ the corresponding maximal ideal in $\mathcal{Z}(\mathfrak{g})$. Put $\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g})/J_\theta\mathcal{U}(\mathfrak{g})$. Denote by $\mathcal{M}(\mathcal{U}_\theta)$ the category of $\mathcal{U}_\theta$-modules.

For any $\lambda \in \mathfrak{h}^*$, A. Beilinson and J. Bernstein defined a twisted sheaf of differential operators $\mathcal{D}_\lambda$ on X with the property that $\Gamma(X, \mathcal{D}_\lambda) = \mathcal{U}_\theta$ (compare [1], [6]). Denote by $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ the category of quasicoherent $\mathcal{D}_\lambda$-modules on X. They also defined the *localization functor* Δ_λ from $\mathcal{M}(\mathcal{U}_\theta)$ into $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ by the formula

$$\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$$

for a $\mathcal{U}_\theta$-module V.

Let $Q(\Sigma)$ be the root lattice in $\mathfrak{h}^*$. For any $\lambda \in \mathfrak{h}^*$, we denote by W_λ the subgroup of the Weyl group W given by $W_\lambda = \{w \in W \mid w\lambda - \lambda \in Q(\Sigma)\}$. Let $\Sigma\check{}$ be the root system in $\mathfrak{h}$ dual to Σ; and for any $\alpha \in \Sigma$, we denote by $\alpha\check{} \in \Sigma\check{}$ the dual root of α. Then, it is well-known that W_λ is the Weyl group of the root system $\Sigma_\lambda = \{\alpha \in \Sigma \mid \alpha\check{}(\lambda) \in \mathbb{Z}\}$. We define an order on Σ_λ by putting $\Sigma_\lambda^+ = \Sigma^+ \cap \Sigma_\lambda$. This defines a set of simple roots Π_λ of Σ_λ, and the corresponding set of simple reflections S_λ. Let ℓ_λ be the length function on (W_λ, S_λ). We say that $\lambda \in \mathfrak{h}^*$ is *regular* if $\alpha\check{}(\lambda)$ is different from zero for any $\alpha \in \Sigma$ and

1991 *Mathematics Subject Classification*. 22E47.

Supported in part by NSF Grant DMS 88-02827

that λ is *antidominant* if $\alpha\check{}(\lambda)$ is not a strictly positive integer for any $\alpha \in \Sigma^+$. We put $n(\lambda) = \min\{\ell_\lambda(w) \mid w \in W_\lambda,\ w\lambda \text{ is antidominant}\}$. Beilinson and Bernstein proved that, for *regular* λ, the left cohomological dimension of the localization functor is $\leq n(\lambda)$ ([2], [8]). In this note we prove the following complementary result.

Theorem. *Let $\lambda \in \mathfrak{h}^*$ be singular. Then the left cohomological dimension of the localization functor Δ_λ is infinite.*

Using the fact that the localization functor is an equivalence of the category $\mathcal{M}(\mathcal{U}_\theta)$ with the category $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ for regular antidominant λ, Beilinson and Bernstein also proved that the homological dimension of $\mathcal{U}_\theta$ is $\leq \frac{1}{2}(\mathrm{Card}(\Sigma)+\mathrm{Card}(\Sigma_\lambda))$ if $\theta = W\cdot\lambda$ consists of regular elements (unpublished, compare [8]). On the contrary, our result immediately implies the following consequence.

Corollary. *If θ consists of singular elements in $\mathfrak{h}^*$, the homological dimension of $\mathcal{U}_\theta$ is infinite.*

This fact was observed earlier by A. Joseph and J. T. Stafford ([7], 4.20). Our argument shows that this is a simple consequence of the analogous behavior of homological dimension for local rings.

Proof of the theorem. Let x be a point in X and denote by $\mathfrak{b}_x$ the corresponding Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ be its nilpotent radical. Let $\mathfrak{h}_x = \mathfrak{b}_x/\mathfrak{n}_x$. Then $\mathfrak{h}_x$ is canonically isomorphic to $\mathfrak{h}$ [6]. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_x$. Then the composition of the projection $\mathfrak{c} \to \mathfrak{h}_x$ with this map gives an isomorphism of $\mathfrak{c}$ onto $\mathfrak{h}$. The inverse map is called a *specialization* at x. For a $\mathcal{U}(\mathfrak{g})$-module V, we put

$$V_{\mathfrak{n}_x} = V/\mathfrak{n}_x V = \mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}_x)} V,$$

where we view $\mathbb{C}$ as a module with the trivial action of $\mathfrak{b}_x$. It has a natural structure of an $\mathfrak{h}_x$-module. Therefore, we can view it as an $\mathfrak{h}$-module. It follows that $V \longrightarrow V_{\mathfrak{n}_x}$ is a right exact covariant functor from the category of $\mathcal{U}(\mathfrak{g})$-modules into the category of $\mathcal{U}(\mathfrak{h})$-modules. If we compose it with the forgetful functor into the category of vector spaces, we get the functor $H_0(\mathfrak{n}_x, -)$ of zeroth $\mathfrak{n}_x$-homology. By the Poincaré-Birkhoff-Witt theorem, free $\mathcal{U}(\mathfrak{g})$-modules are also $\mathcal{U}(\mathfrak{n}_x)$-free, what implies the equality for the left derived functors. Therefore, with some abuse of language, we view the $(-p)^{\text{th}}$ left derived functor of $V \longrightarrow V_{\mathfrak{n}_x}$ as the p^{th} $\mathfrak{n}_x$-homology functor $H_p(\mathfrak{n}_x, -) = \mathrm{Tor}_p^{\mathcal{U}(\mathfrak{n}_x)}(\mathbb{C}, -)$.

We need a technical result, which must be well-known, but we were unable to find a reference.

1. Lemma. *$\mathcal{U}_\theta$ is free as $\mathcal{U}(\mathfrak{n}_x)$-module.*

Proof. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_x$. This determines a specialization of $\mathfrak{h}$ to $\mathfrak{c}$ and a nilpotent subalgebra $\bar{\mathfrak{n}}$ opposite to $\mathfrak{n}_x$. Then we have $\mathfrak{g} = \mathfrak{n}_x \oplus \mathfrak{c} \oplus \bar{\mathfrak{n}}$ and $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}})$ as a left $\mathcal{U}(\mathfrak{n}_x)$-module for left multiplication. Let $\mathrm{F}_p\,\mathcal{U}(\mathfrak{c})$,

$p \in \mathbb{Z}_+$, be the degree filtration of $\mathcal{U}(\mathfrak{c})$. Then we define a filtration $\mathrm{F}_p\,\mathcal{U}(\mathfrak{g})$, $p \in \mathbb{Z}_+$, of $\mathcal{U}(\mathfrak{g})$ via

$$\mathrm{F}_p\,\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} \mathrm{F}_p\,\mathcal{U}(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}).$$

This is clearly a $\mathcal{U}(\mathfrak{n}_x)$-module filtration. The corresponding graded module is

$$\mathrm{Gr}\,\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} S(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}).$$

This filtration induces a filtration on the submodule $J_\theta\mathcal{U}(\mathfrak{g})$ and the quotient module $\mathcal{U}_\theta$. The Harish-Chandra homomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ is compatible with the degree filtrations and the homomorphism $\mathrm{Gr}\,\gamma$ is an isomorphism of $\mathrm{Gr}\,\mathcal{Z}(\mathfrak{g})$ onto the subalgebra $I(\mathfrak{h})$ of all W-invariants in $S(\mathfrak{h})$ ([4], Ch. VIII, §8, no. 5). Denote by $I_+(\mathfrak{h})$ the homogeneous ideal spanned by the elements of strictly positive degree in $I(\mathfrak{h})$. Then

$$\mathrm{Gr}\,J_\theta\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} I_+(\mathfrak{c})S(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}).$$

It follows that

$$\begin{aligned}\mathrm{Gr}\,\mathcal{U}_\theta &= (\mathrm{Gr}\,\mathcal{U}(\mathfrak{g}))/(\mathrm{Gr}\,J_\theta\mathcal{U}(\mathfrak{g})) \\ &= (\mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} S(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}))/(\mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} I_+(\mathfrak{c})S(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}})) \\ &= \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} (S(\mathfrak{c})/(I_+(\mathfrak{c})S(\mathfrak{c}))) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}),\end{aligned}$$

i. e. it is a free $\mathcal{U}(\mathfrak{n}_x)$-module. Moreover, by ([4], Ch. V, §5, no. 2, Th. 1) we know that the dimension of the complex vector space $S(\mathfrak{h})/(I_+(\mathfrak{h})S(\mathfrak{h}))$ is $\mathrm{Card}\,W$. It follows that $\mathcal{U}_\theta$ has a finite filtration by $\mathcal{U}(\mathfrak{n}_x)$-submodules such that $\mathrm{Gr}\,\mathcal{U}_\theta$ is a free $\mathcal{U}(\mathfrak{n}_x)$-module. By induction in length, this implies that $\mathcal{U}_\theta$ is a free $\mathcal{U}(\mathfrak{n}_x)$-module. □

Let ρ be the half-sum of roots in Σ^+. Denote by $\varphi : \mathcal{U}(\mathfrak{h}) \longrightarrow \mathcal{U}(\mathfrak{h})$ the automorphism given by $\varphi(\xi) = \xi + \rho(\xi)$ for $\xi \in \mathfrak{h}$. Then, $\varphi(\gamma(\mathcal{Z}(\mathfrak{g})))$ is the algebra of W-invariants in $\mathcal{U}(\mathfrak{h})$. In addition, as we remarked in the preceding proof, the dimension of the vector space $\mathcal{U}(\mathfrak{h})/\varphi(\gamma(J_\theta))\mathcal{U}(\mathfrak{h})$ is equal to $\mathrm{Card}\,W$. This implies that $V_\theta = \mathcal{U}(\mathfrak{h})/\gamma(J_\theta)\mathcal{U}(\mathfrak{h})$ is an $\mathcal{U}(\mathfrak{h})$-module of dimension $\dim_{\mathbb{C}} V_\theta = \mathrm{Card}\,W$.

For $\mu \in \mathfrak{h}^*$, we denote by I_μ the corresponding maximal ideal in $\mathcal{U}(\mathfrak{h})$.

2. Lemma. *Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$. Then:*

(i) V_θ is a $\mathcal{U}(\mathfrak{h})$-module of dimension $\dim_{\mathbb{C}} V_\theta = \mathrm{Card}\,W$*,*

(ii) the characteristic polynomial of the action of $\xi \in \mathfrak{h}$ on V_θ is

$$P(\xi) = \prod_{w \in W} (\xi - (w\lambda + \rho)(\xi));$$

(iii) $H_0(\mathfrak{n}_x, \mathcal{U}_\theta)$ is a direct sum of countably many copies of V_θ.

Proof. We already established (i). Clearly, $I_\mu \supset \varphi(\gamma(J_\theta))\mathcal{U}(\mathfrak{h})$ is equivalent to $\mu = w\lambda$ for some $w \in W$. Hence the linear transformation of $\mathcal{U}(\mathfrak{h})/\varphi(\gamma(J_\theta))\mathcal{U}(\mathfrak{h})$ induced by multiplication by ξ has eigenvalues $(w\lambda)(\xi)$, $w \in W$, and by symmetry they all have the same multiplicity. This in turn implies that

$$\varphi(P(\xi)) = \prod_{w\in W} \varphi(\xi - (w\lambda + \rho)(\xi)) = \prod_{w\in W} (\xi - (w\lambda)(\xi))$$

is the characteristic polynomial for the action of ξ on $\mathcal{U}(\mathfrak{h})/\varphi(\gamma(J_\theta))\,\mathcal{U}(\mathfrak{h})$. This proves (ii).

(iii) As in the proof of 1, we fix a specialization $\mathfrak{c}$ of $\mathfrak{h}$ and choose a nilpotent subalgebra $\bar{\mathfrak{n}}$ opposite to $\mathfrak{n}_x$. By Poincaré-Birkhoff-Witt theorem, it follows that as a vector space $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}})$. Moreover,

$$H_0(\mathfrak{n}_x, \mathcal{U}_\theta) = \mathcal{U}(\mathfrak{g})/(J_\theta\, \mathcal{U}(\mathfrak{g}) + \mathfrak{n}_x\, \mathcal{U}(\mathfrak{g})).$$

Denote by $\gamma_x : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{c})$ the composition of the specialization map with the Harish-Chandra homomorphism γ. Then

$$J_\theta\, \mathcal{U}(\mathfrak{g}) + \mathfrak{n}_x\, \mathcal{U}(\mathfrak{g}) = J_\theta\, \mathcal{U}(\mathfrak{c})\,\mathcal{U}(\bar{\mathfrak{n}}) + \mathfrak{n}_x\, \mathcal{U}(\mathfrak{g}) = \gamma_x(J_\theta)\,\mathcal{U}(\mathfrak{c})\,\mathcal{U}(\bar{\mathfrak{n}}) + \mathfrak{n}_x\, \mathcal{U}(\mathfrak{g}),$$

which implies that under the above isomorphism

$$J_\theta\, \mathcal{U}(\mathfrak{g}) + \mathfrak{n}_x\, \mathcal{U}(\mathfrak{g}) = \big(\mathbb{C} \otimes_{\mathbb{C}} \gamma_x(J_\theta)\,\mathcal{U}(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}})\big) \oplus \big(\mathfrak{n}_x \mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{c}) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}})\big).$$

This yields

$$H_0(\mathfrak{n}_x, \mathcal{U}_\theta) = \mathcal{U}(\mathfrak{c})/(\gamma_x(J_\theta)\,\mathcal{U}(\mathfrak{c})) \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}}) = V_\theta \otimes_{\mathbb{C}} \mathcal{U}(\bar{\mathfrak{n}})$$

and the action of $\mathfrak{h}$ is given by multiplication in the first factor. □

Therefore, maximal ideals in the ring V_θ are the images of the maximal ideals $I_{w\lambda+\rho}$, $w \in W$, in $\mathcal{U}(\mathfrak{h})$, under the quotient map $\mathcal{U}(\mathfrak{h}) \longrightarrow V_\theta$.

Let $W(\lambda)$ be the stabilizer of λ in W. Denote by $R_{w\lambda}$ the localization of V_θ at $I_{w\lambda+\rho}$. Then, by ([3], Ch. IV, §2, no. 5, Cor. 1 of Prop. 9), we have

$$V_\theta = \prod_{w\in W/W(\lambda)} R_{w\lambda}.$$

Since the local rings $R_{w\lambda}$ are finite-dimensional, they are regular if and only if $\dim_{\mathbb{C}} R_{w\lambda} = 1$. Because of the Weyl group symmetry these rings are isomorphic, hence $\dim_{\mathbb{C}} R_{w\lambda} = \operatorname{Card} W(\lambda)$ for any $w \in W$. This finally leads to the following critical observation.

3. Corollary. *The following conditions are equivalent:*

(i) λ is regular;

(ii) the rings $R_{w\lambda}$, $w \in W$, are regular local rings.

By 1, we can calculate $\mathfrak{n}_x$-homology of V using a left resolution of V by free $\mathcal{U}_\theta$-modules. Therefore, we can view $H_p(\mathfrak{n}_x, V)$ as V_θ-modules. Also, for any $\lambda \in \theta$, $\mathbb{C}_{\lambda+\rho} = \mathcal{U}(\mathfrak{h})/I_{\lambda+\rho}$ is a V_θ-module.

For any $\mathcal{O}_X$-module $\mathcal{F}$ on X we denote by $T_x(\mathcal{F})$ its geometric fibre at x, i. e. $T_x(\mathcal{F}) = \mathbb{C} \otimes_{\mathcal{O}_x} \mathcal{F}_x$, where $\mathcal{O}_x$ is the local ring of X at x. Since X is a smooth algebraic variety, $\mathcal{O}_x$ is a regular local ring. Hence, the left cohomological dimension of the right exact functor T_x is $\leq \dim X$.

For any abelian category $\mathcal{A}$, denote by $D^-(\mathcal{A})$ the derived category of $\mathcal{A}$-complexes bounded from above, and by D the natural imbedding of $\mathcal{A}$ into $D^-(\mathcal{A})$ which maps an object V of $\mathcal{A}$ into the complex $D(V)$ such that $D(V)^p = 0$ for $p \neq 0$ and $D(V)^0 = V$.

Since the localization functor Δ_λ is right exact, it defines the left derived functor $L\Delta_\lambda$ from $D^-(\mathcal{U}_\theta)$ into $D^-(\mathcal{D}_\lambda)$. Analogously, T_x defines the left derived functor LT_x from $D^-(\mathcal{D}_\lambda)$ into the derived category $D^-(\mathbb{C})$ of complexes of complex vector spaces bounded from above.

4. Proposition. *Let $\lambda \in \mathfrak{h}^*$, $\theta = W \cdot \lambda$ and $x \in X$. Then the functors $LT_x \circ L\Delta_\lambda$ and $D(\mathbb{C}_{\lambda+\rho}) \overset{L}{\otimes}_{V_\theta} (D(\mathbb{C}) \overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)} -)$ from $D^-(\mathcal{U}_\theta)$ into $D^-(\mathbb{C})$ are isomorphic.*

Proof. By 1, we know that $\mathcal{U}_\theta$ is acyclic for the functor $H_0(\mathfrak{n}_x, -) = \mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}_x)} -$. By 2, we also know that $\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}_x)} \mathcal{U}_\theta$ is acyclic for the functor $\mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} -$. Let $F^\cdot$ be a complex isomorphic to $V^\cdot$ consisting of free $\mathcal{U}_\theta$-modules. Then, since the functors commute with infinite direct sums, we get

$$D(\mathbb{C}_{\lambda+\rho}) \overset{L}{\otimes}_{V_\theta} (D(\mathbb{C}) \overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)} V^\cdot) = \mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}_x)} F^\cdot).$$

On the other hand, the localization $\Delta_\lambda(\mathcal{U}_\theta) = \mathcal{D}_\lambda$ is a locally free $\mathcal{O}_X$-module, and therefore acyclic for T_x. This implies that

$$LT_x(L\Delta_\lambda(V^\cdot)) = T_x(\Delta_\lambda(F^\cdot)).$$

Hence, to complete the proof it is enough to establish the following identity

$$T_x(\Delta_\lambda(\mathcal{U}_\theta)) = \mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}_x)} \mathcal{U}_\theta).$$

First, we have $T_x(\Delta_\lambda(\mathcal{U}_\theta)) = T_x(\mathcal{D}_\lambda)$. Moreover, from the construction of $\mathcal{D}_\lambda$ ([1], [6]) and the properties of the Harish-Chandra homomorphism, it follows that

$$\begin{aligned} T_x(\mathcal{D}_\lambda) &= (\mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g}))/(I_{\lambda+\rho}(\mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g}))) \\ &= \mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g}))/(\gamma(J_\theta)(\mathcal{U}(\mathfrak{g})/\mathfrak{n}_x\mathcal{U}(\mathfrak{g}))) \\ &= \mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} (\mathcal{U}(\mathfrak{g})/(J_\theta\mathcal{U}(\mathfrak{g}) + \mathfrak{n}_x\mathcal{U}(\mathfrak{g}))) = \mathbb{C}_{\lambda+\rho} \otimes_{V_\theta} H_0(\mathfrak{n}_x, \mathcal{U}_\theta). \quad \square \end{aligned}$$

5. Remark. Using spectral sequences instead of derived categories, 4. implies the following statement: The Grothendieck spectral sequences for composition of derived functors with E_2-terms $E_2^{p,q} = L^pT_x(L^q\Delta_\lambda(V))$ and $E'^{p,q}_2 = \mathrm{Tor}^{V_\theta}_{-p}(\mathbb{C}_{\lambda+\rho}, H_{-q}(\mathfrak{n}_x, V))$ converge to the same limit.

To prove the theorem it is enough to establish the following fact.

6. Lemma. *Let $\lambda \in \mathfrak{h}^*$ be singular. Then there exists $V \in \mathcal{M}(\mathcal{U}_\theta)$ such that $L\Delta_\lambda(D(V))$ is not a cohomologically bounded complex.*

Proof. Since the functor T_x has finite left cohomological dimension, it is enough to find a $\mathcal{U}_\theta$-module V such that $LT_x(L\Delta_\lambda(V))$ is not a bounded complex for some $x \in X$. By 4, this is equivalent to the fact that $D(\mathbb{C}_{\lambda+\rho})\overset{L}{\otimes}_{V_\theta}(D(\mathbb{C})\overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)}D(V))$ is not a cohomologically bounded complex.

Let w_0 be the longest element in W. Fix a Borel subalgebra $\mathfrak{b}_0$, and consider the Verma module $M(w_0\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_0)} \mathbb{C}_{w_0\lambda-\rho}$. Pick x so that $\mathfrak{b}_x$ is opposite to $\mathfrak{b}_0$. Then, by Poincaré-Birkhoff-Witt theorem, $M(w_0\lambda)$ is isomorphic to $\mathcal{U}(\mathfrak{n}_x) \otimes_{\mathbb{C}} \mathbb{C}_{w_0\lambda-\rho}$ as $\mathcal{U}(\mathfrak{n}_x)$-module. This implies, since $\mathfrak{b}_x$ is opposite to $\mathfrak{b}_0$ and corresponding specializations differ by w_0, that

$$H_0(\mathfrak{n}_x, M(w_0\lambda)) = \mathbb{C}_{\lambda+\rho},$$

and $H_p(\mathfrak{n}_x, M(w_0\lambda)) = 0$ for $p \in \mathbb{N}$. Therefore,

$$D(\mathbb{C})\overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)}D(M(w_0\lambda)) = D(\mathbb{C}_{\lambda+\rho}),$$

and

$$D(\mathbb{C}_{\lambda+\rho})\overset{L}{\otimes}_{V_\theta}(D(\mathbb{C})\overset{L}{\otimes}_{\mathcal{U}(\mathfrak{n}_x)}D(M(w_0\lambda))) = D(\mathbb{C}_{\lambda+\rho})\overset{L}{\otimes}_{V_\theta}D(\mathbb{C}_{\lambda+\rho}).$$

Clearly, we have

$$H^{-p}(D(\mathbb{C}_{\lambda+\rho})\overset{L}{\otimes}_{V_\theta}D(\mathbb{C}_{\lambda+\rho})) = \operatorname{Tor}_p^{V_\theta}(\mathbb{C}_{\lambda+\rho}, \mathbb{C}_{\lambda+\rho}),\ p \in \mathbb{Z}_+.$$

On the other hand, we have

$$\operatorname{Tor}_p^{V_\theta}(\mathbb{C}_{\lambda+\rho}, \mathbb{C}_{\lambda+\rho}) = \operatorname{Tor}_p^{R_\lambda}(\mathbb{C}, \mathbb{C}),\ p \in \mathbb{Z}_+.$$

Since R_λ is not a regular local ring by 3, its homological dimension is infinite ([5], 17.3.1) and $\operatorname{Tor}_p^{R_\lambda}(\mathbb{C}, \mathbb{C}) \neq 0$ for $p \in \mathbb{Z}_+$ ([5], 17.2.11). ☐

This completes the proof of the theorem.

References

1. A. Beilinson, J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris, Ser. I **292** (1981), 15–18.
2. A. Beilinson, J. Bernstein, *A generalization of Casselman's submodule theorem*, Representation Theory of Reductive Groups, Birkhäuser, Boston, 1983, pp. 35–52.
3. N. Bourbaki, *Algèbre commutative*, Masson, Paris.
4. N. Bourbaki, *Groupes et algèbres de Lie*, Masson, Paris.
5. A. Grothendieck, *Eléments de Géométrie Algébrique IV*, Publ. I.H.E.S. **No. 20** (1964).
6. H. Hecht, D. Miličić, W. Schmid, J. A. Wolf, *Localization and standard modules for real semisimple Lie groups I: The duality theorem*, Inventiones Math. **90** (1987), 297–332.
7. A. Joseph, J. T. Stafford, *Modules of $\mathfrak{k}$-finite vectors over semi-simple Lie algebras*, Proc. London Math. Soc. **49** (1984), 361–384.
8. D. Miličić, *Localization and representation theory of reductive Lie groups*, (mimeographed notes), to appear.

Department of Mathematics, University of Utah, Salt Lake City, UT 84112