The Weil-Petersson Metric on Teichmüller and Moduli Space

Ken Bromberg

Notes written by
Yael Algom-Kfir,
William Malone,
Erika Meucci

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Chapter 1

The Hyperbolic Plane, Models and Isometries

There are many ways to define Hyperbolic Geometry. We will present two viewpoints. In the Geometric Viewpoint, we start from the definition of isometries of the Hyperbolic Plane and then we find a metric for such space. Conversely, in the Analytic Viewpoint, first we define the metric in the Hyperbolic Plane and then we find the isometries of this metric space.

1.1 Geometric Viewpoint

In this section we will use a construction from Euclidean geometry to introduce “straight” lines for hyperbolic geometry as well as a group of diffeomorphisms that act nicely on these lines. Finally, we will derive the Riemannian metric invariant under this set of diffeomorphisms. This take on Hyperbolic Geometry is from Thurston’s Book.

Fact 1.1.1. (From Euclidean Geometry) $\vec{PA} \cdot \vec{PB}$ (Euclidean dot product) is independent of the choice of the line. In the following picture this says that $\vec{PA} \cdot \vec{PB} = P t^2$.

Figure 1.1: $\vec{PA} \cdot \vec{PB} = P t^2$. 
CHAPTER 1. THE HYPERBOLIC PLANE, MODELS AND ISOMETRIES

Proof.

\[ \vec{PA} = \vec{PM} + \vec{MA} \]
\[ \vec{PB} = \vec{PM} + \vec{MB} \]
\[ = \vec{PM} - \vec{MA} \]
\[ \vec{PA} \cdot \vec{PB} = |PM|^2 - |MA|^2 \]
\[ = (|PO|^2 - |OM|^2) - (|OB|^2 - |OM|^2) \text{ (Pythagorean theorem)} \]
\[ = |PO|^2 - |OB|^2 \]
\[ = |PO|^2 - \text{radius}^2 \]

which is independent of \( A \) and \( B \). \( \square \)

1.1.1 Inversions in Circles

Let \( c \) be a circle such that \( c \subset \mathbb{R}^2 \cup \infty = \text{Riemann Sphere} = \hat{\mathbb{C}} \). Define \( i_c : \mathbb{R}^2 \cup \infty \rightarrow \mathbb{R}^2 \cup \infty \) via the following:

\[ O \]
\[ P \]
\[ P' \]
\[ O \]
\[ c \]
\[ \mathbb{R} \]
\[ P' \]
\[ O \]
\[ P \]

\[ \text{Figure 1.2: Definition of } i_c. \]

such that \( OP \cdot OP' = R^2 \). Define \( i_c(P) = P', i_c(0) = \infty, i_c(\infty) = 0. \)

Properties:

0) \( i_c \) is a diffeomorphism.

1) \( i_c|_c = id. \)

2) \( i_c \) interchanges the inside and outside of the circle.

3) Lines through the origin \((\cup \infty)\) are invariant under \( i_c. \)

4) Circles orthogonal to \( c \) are \( i_c \)-invariant.

5) If \( h : \mathbb{R}^2 \cup \infty \rightarrow \mathbb{R}^2 \cup \infty \) is a homothety \((x \rightarrow \lambda x) \ (\lambda > 0)\), then \( i_c h = h^{-1}i_c. \)

The only property that is not obvious is number 4.
Proof. (of 4.) Since the two circles \( c, c' \) are orthogonal, there exists a point \( t \in c \cap c' \) and the tangent line at \( t \) to \( c' \) goes through \( O \). The fact then implies \( \overrightarrow{OP} \cdot \overrightarrow{OP'} = |\overrightarrow{OT}|^2 = R^2 \). Thus \( i_c(P) = P' \).

![Figure 1.3: Circles orthogonal to \( c \) are \( i_c \)-invariant.](image)

More Properties:

1. \( i_c \) is conformal (preserves angles).

2. Circles not containing 0 go to other such circles.

3. Circles through 0 are interchanged with lines(\( \cup \infty \)) not through 0.

Proof. Of 1. Find circles \( c_u \) and \( c_v \) that are \( \perp c \) and tangent to \( u \) and \( v \) respectively.

![Figure 1.4: \( i_c \) is conformal.](image)

Since \( i_c \) preserves \( c_v \) and \( c_u \), \( i_c(P) = P' = (\text{other intersection point}) \). \( i_{c_u}(u) = u' \) and \( i_{c_v}(v) = v' \) where \( u', v' \) are tangent to \( i_u, i_v \) at \( P' \). Thus \( \angle(u, v) = \angle(u', v') \).
Figure 1.5: Circles not containing 0 go to other such circles.

Of 2. Start by showing that \{circles, lines\} = clines tangent to \(c\) map to clines of the same form. Consider the family \(\mathcal{F}_t\) of clines tangent to \(c\) at a point \(p\). Also let \(\mathcal{F}_p\) be the family of clines perpendicular to \(c\) at \(p\).

When a curve from \(\mathcal{F}_t\) intersects \(\mathcal{F}_p\), they meet perpendicularly. \(i_c\) preserves \(\mathcal{F}_t\), so it preserves the line field perpendicular to \(\mathcal{F}_t\). So \(i_c\) preserves the corresponding integral manifolds, and these are \(\mathcal{F}_t\).

Now let \(c'\) be any circle not through 0. There is a homothety \(h\) such that \(h(c')\) is tangent to \(c\). Then \(i_c(c') = i_c(h^{-1}[h(c')]) = hi_c(h(c'))\) where \(i_c(h(c'))\) is a circle tangent to \(c\).

Of 3. Mimic the proof of 2.

\[\square\]

1.1.2 Poincaré Model

Let \(\Delta\) be the open unit disc in \(\mathbb{R}^2\). A geodesic is a diameter or an arc of a circle perpendicular to \(\partial \Delta\).

A reflection \(r_\gamma : \Delta \to \Delta\) in a geodesic \(\gamma\) is the restriction to \(\Delta\) of the inversion \(i_c\) where \(\gamma \subset c\) or the euclidean reflection when \(\gamma\) is a straight line.

An “isometry” of \(\Delta\) is a composition of reflections. These naturally form a group to which we associate the following notation:

\[Isom(\Delta) = \text{group of “isometries” of } \Delta\]

\[Isom^+(\Delta) = \text{index 2 subgroup consisting of orientation preserving “isometries”}\]
Properties. 1. $Isom^+(\Delta)$ acts transitively on $\Delta$.

2. Any isometry takes geodesics to geodesics and $Isom^+(\Delta)$ acts transitively on the set of all geodesics.

3. The stabilizer of $0 \in \Delta$ consists of the restrictions to $\Delta$ of Euclidean isometries fixing $0$.

4. Any two points in $\Delta$ are on a unique geodesic. (Postpone until we know the metric)

Proof. Of 1. Use the intermediate value theorem on geodesic segments that meet the geodesic from $0$ to $P$ perpendicularly and then compose with a reflection through $0$ and $P$.

Of 2. The first part follows from the properties of inversions. For the second part we can assume without loss of generality that one of the points is $0$ by conjugation by the isometry from the first property. Now since we can find an isometry that takes $P$ to $0$ by property one and the first part says that this is a geodesic
through 0 (i.e., a diameter). Then rotate (composition of two reflections) to land on the desired geodesic.

Of 3. $O(2) \subset Isom(\Delta)$ and $O(2) = \{\text{group of isometries fixing origin}\}$. $O(2)$ is topologically the wedge of two circles.

Let $\varphi \in Isom(\Delta)$ fix 0. Compose $\varphi$ (if necessary) with an element of $O(2)$ so that $\varphi$ is orientation-preserving and leaves a ray from 0 invariant.

**Claim 1.1.2.** $\varphi = id$

*Proof.* (Of Claim) Since $\varphi$ preserves angles, it preserves all rays through 0. This implies that all geodesics are preserved. Now given any point represent it as the intersection of two geodesics so that every point is fixed. □

The claim implies that $Stab(0) = O(2)$. □

### 1.1.3 Deriving the Riemannian metric on $\Delta$

**Definition 1.1.3.** Let $\rho_1$ and $\rho_2$ be two Riemannian metrics on $X$. We say that $\rho_1$ is *conformally equivalent* to $\rho_2$ if there exists a continuous map $\phi : X \to \mathbb{R}^+$ such that for all $x \in X$:

$$<v_x, w_x>_{\rho_1} = \phi(x) <v_x, w_x>_{\rho_2}.$$ 

This is equivalent to the condition that the angles measured with respect to the two metrics are equal.

**Corollary 1.1.4.** Up to scale, there is a unique Riemannian metric on $\Delta$ which is invariant under the isometry group. This metric is conformally equivalent to the Euclidean metric. Geodesics in the Riemannian metric are Euclidean lines and circles perpendicular to $\partial\Delta$ intersected with $\Delta$. Circles in the Riemannian metric (i.e., the locus of points equidistant from the center) are Euclidean circles.
1.1. GEOMETRIC VIEWPOINT

**Figure 1.9:** Because in a Riemannian metric space local geodesics are unique, a geodesic must be contained in the fixed point set of an isometry that fixes $x$ and $y$.

**Proof.** Let $\rho$ be a Riemannian metric invariant under the isometry group. Notice that $\rho$ is uniquely determined by its value at a base point (say $0 \in \Delta$) since the group acts transitively on $X$. After rescaling we may assume that $\rho$ is equal the Euclidean Riemannian metric at that base point. Now at 0, Euclidean angles and $\rho$ angles coincide. Since inversions preserve Euclidean angles, $\rho$-angles and Euclidean angles agree at every point. Indeed, if $v, w \in T_x\Delta$ let $g$ be an isometry taking $x$ to 0, then

$$\angle_\rho(v, w) = \angle_\rho(d_xg(v), d_xg(w))$$

where the later two vectors lie in $T_0\Delta$. Hence,

$$\angle_\rho(d_xg(v), d_xg(w)) = \angle_{\text{Euc}}(d_xg(v), d_xg(w)) = \angle_{\text{Euc}}(d_0g^{-1}(d_xg(v)), d_0g^{-1}(d_xg(w))) = \angle_{\text{Euc}}(v, w).$$

Thus, $\rho$ and Euc are conformally equivalent.

Since rotations about the origin are isometries for both the Euclidean and the hyperbolic metrics on $\Delta$, Euclidean circles about the origin are hyperbolic circles. Since Euclidean circles are preserved by hyperbolic isometries, they are hyperbolic circles even when they are no longer centered at the origin (however, the Euclidean and hyperbolic centers of the circle do not coincide).

Let $\gamma$ be a hyperbolic geodesic from $x$ to $y$. Then $x$ and $y$ either lie on a circle perpendicular to $\partial\Delta$ or on a diameter of $\Delta$. Let $C$ be the diameter or arc of the circle intersected with $\Delta$. Then $g = i_C$ is an isometry fixing $x$ and $y$. Therefore, it takes $\gamma$ to another hyperbolic geodesic from $x$ to $y$. In a Riemannian metric, geodesics are locally unique. Therefore, if $x$ and $y$ are sufficiently close, $g(\gamma) = \gamma$ so $\gamma$ must be the subarc of $C$ between $x$ and $y$ (see Figure 1.9). Now even if $x$ and $y$ aren’t close, $\gamma$ must be a local geodesic, so at every point it must coincide with a subarc of a diameter or a circle perpendicular to $\partial\Delta$. But two such distinct curves cannot be
CHAPTER 1. THE HYPERBOLIC PLANE, MODELS AND ISOMETRIES

Figure 1.10: Geodesics in the upper half space model are semicircles centered on the real line, and vertical lines.

tangent at a point in $\Delta$ and so a hyperbolic geodesic must be a subarc of a diameter of $\Delta$ or a circle perpendicular to $\partial \Delta$.

**Question 1.1.5.** Write down an explicit expression for the metric.

### 1.1.4 The Upper Half Space Model

We define

$$\mathbb{H}^2 := \{ z \in \mathbb{C} | \text{Im} z > 0 \}.$$ 

One can apply an inversion to $\Delta$ to obtain $\mathbb{H}^2$ (for example: $\phi(z) = \frac{1-z}{1-z}$).

The isometry group of $\mathbb{H}^2$ consists of reflections about these geodesics.

**Theorem 1.1.6.** The following map is an isomorphism:

$$\Phi : \text{PSL}_2(\mathbb{R}) \rightarrow \text{Isom}^+(\mathbb{H}^2).$$

$$(a \ b \ c \ d) \mapsto (z \mapsto \frac{az+b}{cz+d})$$

**Proof.** First consider the above map with $\text{SL}_2(\mathbb{R})$ as its domain. We must show (among other things) that it lands where it’s supposed to - orientation preserving isometries. Consider the images of the following matrices:

- $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \rightarrow (z \rightarrow z+r)$ this is a composition of two reflections in geodesic lines perpendicular to $\mathbb{R}$. Thus it is an orientation-preserving isometry.

- $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow (z \rightarrow \frac{1}{z})$ this is a composition of reflection in the unit circle: $\frac{1}{z}$ and in the y-axis: $-z$. Thus it is an orientation-preserving isometry.

Notice that $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ therefore lower triangular matrices also land in orientation-preserving isometries. In addition, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ is sent to the map $z \rightarrow \lambda^2 z$ which is a composition of two inversions centered at the origin.
1.1. GEOMETRIC VIEWPOINT

Figure 1.11: If $\phi$ doesn’t stabilize $i$, compose with maps in the image of $\Phi$ to get one which does.

(much like the composition of two reflections in perpendicular lines is a translation). Finally, by Gaussian elimination, every matrix in $\text{SL}_2(\mathbb{R})$ can be written as a product of

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$$

hence the map on $\text{SL}_2(\mathbb{R})$ is well defined. Next, a matrix is in the kernel of this map if $b = c = 0$ and $a = d$. Thus this map descends to a monomorphism from $\text{PSL}_2(\mathbb{R})$.

We wish to compute which matrices land in $\text{Stab}(i)$. Suppose $\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Stab}(i)$, then $\frac{ai + b}{ci + d} = i$ which implies $a = d, b = -c$. Since the determinant equals 1, we get $a^2 + b^2 = 1$ and therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSO}(2) = \text{SO}(2)$. Therefore, $\Phi$ maps $\text{SO}(2)$ onto the stabilizer of $i$ in $\text{Isom}^+(H)$. Now let $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ be any orientation-preserving isometry. After composing $\phi$ with a translation and a dilation (which are both in the image of $\Phi$) we can assume that $\phi$ fixes $i$, and we’ve already shown that such a map is in the image.

Now we are in a better position to figure out what the invariant Riemannian metric should be. At $i$, choose the metric to coincide with the standard inner product. Since horizontal translations are isometries, the metric should only be a function of $y$. Since dilation by $\lambda$ is an isometry the vector $v$ at $i$ and the vector $\lambda v$ at $\lambda i$ should have the same norm. Thus we obtain the metric $\sqrt{\frac{1}{y^2}}dt^2$ where $dt^2$ is the Euclidean metric.

**Exercise 1.1.7.** In $\Delta$, the hyperbolic metric is: $ds^2 = \frac{4}{(1-v^2)^2}dt^2$.

**Remark.** By the formula:

$$4 < v, w > = \|v + w\|^2 - \|v - w\|^2,$$

the norm determines the inner product when it exists.
1.2 Analytic Viewpoint

1.2.1 Models of Hyperbolic Geometry

In the last section we started from the isometries of the Hyperbolic Plane and we deduced its metric. In this section, we start with the metric of the Hyperbolic Plane and we find its isometries.

There are two main models of the hyperbolic geometry: the Upper Half Plane Model and the Poincaré Disc Model.

In the Upper Half Plane Model the metric is

\[ g_{\mathbb{H}^2}(z) = \frac{1}{(Im(z))^2}g_{\mathbb{E}^2}, \]

where \( g_{\mathbb{E}^2} \) denotes the Euclidean metric, and the geodesics are the intersections of \( \mathbb{H}^2 \) with the Euclidean lines in \( \mathbb{C} \) perpendicular to the real axis and the intersections of \( \mathbb{H}^2 \) with Euclidean circles centered on the real axis (see Figure 1.12).

In the Poincaré Disc Model the metric is

\[ g_{\mathbb{D}}(z) = \frac{4}{(1 - |z|^2)^2}g_{\mathbb{E}^2}|_{\text{disc}} \]

and the geodesics are the intersections of \( \mathbb{D} \) with the Euclidean lines in \( \mathbb{C} \) through the origin and the intersections of \( \mathbb{D} \) with Euclidean circles perpendicular to \( \partial \mathbb{D} \) (see Figure 1.13).

1.2.2 Length and Area

We will identify \( z \in \mathbb{C} \) with \( (x, y) \in \mathbb{R}^2 \).

**Definition 1.2.1.** If \( \gamma : [a, b] \to \mathbb{H}^2 \) is a \( C^1 \)-path, we define the length of \( \gamma \)

\[ l_{\mathbb{H}^2}(\gamma) := \int_{\gamma} \frac{1}{Im(z)}|dz| = \int_{a}^{b} \frac{|\gamma'(t)|}{Im(\gamma(t))}dt. \]
For example, if $\gamma : [a, b] \to \mathbb{H}^2$, $\gamma(t) = it$, then
\[
G_{\mathbb{H}^2} (\gamma) = \int_\gamma \frac{1}{Im(z)} |dz| = \int_a^b \frac{|\gamma'(t)|}{Im(\gamma(t))} dt = \int_a^b \frac{1}{t} dt = \ln \left( \frac{b}{a} \right)
\]
since $|\gamma'(t)| = 1$ and $Im(\gamma(t)) = t$.

**Definition 1.2.2.** If $X \subset \mathbb{H}^2$, we define the area of $X$
\[
\text{area}(X) := \int_X \frac{1}{Im(z)^2} dx dy = \int_X \frac{1}{y^2} dx dy.
\]

For example, if $X \subset \mathbb{H}^2$ is the region bounded by the three Euclidean lines
\[
\{z \in \mathbb{H}^2 | Re(z) = 1, Im(z) > 1\}, \{z \in \mathbb{H}^2 | Re(z) = -1, Im(z) > 1\} \text{ and } \{z \in \mathbb{H}^2 | Im(z) = 1\} \text{ (see Figure 1.14)},
\]
then
\[
\text{area}(X) = \int_X \frac{1}{y^2} dx dy = \int_{-1}^{1} \int_{1}^{\infty} \frac{1}{y^2} dy dx = 2.
\]

**1.2.3 Isometries and Uniqueness of Geodesics**

We know the metric of the Hyperbolic Plane and so we can consider the group $Isom^+(\mathbb{H}^2)$ of the orientation preserving isometries of $\mathbb{H}^2$.

**Theorem 1.2.3.** $Isom^+(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$.

**Proof.** (Sketch) We define
\[
\varphi : \ PSL_2(\mathbb{R}) \longrightarrow Isom^+(\mathbb{H}^2).
\]
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (z \longmapsto \frac{az+b}{cz+d})
\]
We have to check that $\varphi$ is a well-defined bijective homomorphism.

We start proving that $\varphi$ is well-defined, i.e. $\gamma(z) = \frac{az + b}{cz + d} \in Isom^+(\mathbb{H}^2)$. In other words, we want to show that

$$\frac{|\gamma'(z)|}{Im(\gamma(z))} = \frac{1}{Im(z)}.$$

Since

$$\gamma'(z) = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2}$$

and

$$\frac{az + b}{cz + d} = \frac{(az + b)(cz + d)}{|cz + d|^2} = \frac{acz + adz + bcz + bd}{|cz + d|^2}, \quad (a, b, c, d \in \mathbb{R})$$

we have

$$|\gamma'(z)| = \frac{ad - bc}{|cz + d|^2} \quad \text{and} \quad Im(\gamma(z)) = \frac{ady - bcy}{|cz + d|^2} = \frac{(ad - bc)y}{|cz + d|^2}.$$

Hence,

$$\frac{|\gamma'(z)|}{Im(\gamma(z))} = \frac{(ad - bc)}{|cz + d|^2} \frac{|cz + d|^2}{(ad - bc)y} = \frac{1}{Im(z)}.$$

Now, since

$$Ker(\varphi) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R}) \mid \frac{az + b}{cz + d} = z \right\} =$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R}) \mid b = c, a = d \right\} = Id \in PSL_2(\mathbb{R}),$$

$\varphi$ is injective.

We leave, as an exercise, to prove that $\varphi$ is a surjective homomorphism. \qed
Using this theorem we can prove the following

**Proposition 1.2.4.** There exists a unique hyperbolic geodesic between two points in $\mathbb{H}^2$.

**Proof.** Without loss of generality, we can suppose that the two points are $ia$ and $ib$. We want to prove that $\gamma : [a, b] \to \mathbb{H}^2$, $\gamma(t) = it$ is the unique geodesic through these two points.

Obviously, $\gamma$ is a geodesic. We need to prove the uniqueness. By contradiction, we suppose that there exists a geodesic $f : [a, b] \to \mathbb{H}^2$ such that $f(a) = ia$, $f(b) = ib$.

![Figure 1.15: The path $\gamma$ and $f$ of the Proposition 1.2.4.](image)

We consider the map $\varphi(re^{i\theta}) = ri$, i.e., $f(x, y) = (0, \sqrt{x^2 + y^2})$. Since $\varphi$ is an orientation preserving isometry, if we prove that $\varphi$ is a contraction, then $l(\varphi(f)) < l(f)$ and so $f$ is not a geodesic.

From this argument it follows that a geodesic between $ia$ and $ib$ is contained in the imaginary axis. Now, we observe that $\gamma$ is the smallest path between $ia$ and $ib$ that is contained in the imaginary axis. Hence, we need to show that $\varphi$ is a contraction and we are done.

We consider a tangent vector $v = (x, y)$. Since the Jacobian of $\varphi$ is

$$D(\varphi) = \begin{pmatrix} 0 & 0 \\ \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \end{pmatrix}$$

and so

$$D(\varphi)(v) = \begin{pmatrix} 0 & 0 \\ \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{x^2 + y^2} \end{pmatrix},$$

we have

$$\|v\|_{(x, y)} = \sqrt{\frac{x^2 + y^2}{y}} = \sqrt{1 + \left(\frac{x}{y}\right)^2} > 1 = \sqrt{\frac{x^2 + y^2}{\sqrt{x^2 + y^2}^2}} = \|D\varphi(v)\|_{(0, \sqrt{x^2 + y^2})}.$$
Since the tangent vectors shrinks, \( \varphi \) is a contraction and this concludes the proof of the proposition. \( \square \)

Since the Hyperbolic Plane is uniquely geodesic, we can define the distance between two points \( p \) and \( q \) as

\[
d_{H^2}(p, q) := \min_\gamma \{l(\gamma) \mid \gamma : [0, 1] \to H^2, \gamma(0) = a, \gamma(1) = b\}.
\]

### 1.3 Classification of Isometries

We can ask the following question: *How many fixed points an orientation preserving isometry has in \( H^2 \)?*

We can compute the fixed points solving

\[
a z + b \overline{c z + d} = z \quad \text{i.e.} \quad c z^2 + (d - a) z - b = 0.
\]

We have two cases. If \( c = 0 \), then we have only one fixed point at \( \infty \) if \( d = a \) and two fixed points \( \infty \) and \( \frac{b}{d-a} \) if \( d \neq a \).

Now, if \( c \neq 0 \), then

\[
z = \frac{-(d-a) \pm \sqrt{(d-a)^2 + 4bc}}{2c}.
\]

Let \( \Delta := (d-a)^2 + 4bc \). Since \( ad - bc = 1 \),

\[
\Delta = (d-a)^2 + 4bc = d^2 + a^2 - 2ad + 4bc = d^2 + a^2 + 2ad - 4 = (d+a)^2 - 4 = trace \begin{pmatrix} a & b \\ c & d \end{pmatrix} - 4.
\]

We have three cases:

1. \( \Delta < 0 \Leftrightarrow \left| trace \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| < 2 \), then we get two conjugate complex roots and hence, we have only one fixed point in \( H^2 \). In this case, the isometry is called *elliptic*;

2. \( \Delta = 0 \Leftrightarrow \left| trace \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = 2 \), then we get one real root and hence, we have only one fixed point in the real axis of \( H^2 \). In this case, the isometry is called *parabolic*;

3. \( \Delta > 0 \Leftrightarrow \left| trace \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| > 2 \), then we get two real roots and hence, we have two fixed point in the real axis of \( H^2 \). In this case, the isometry is called *hyperbolic or loxodromic*.
Since two Möbius transformations $M$ and $N$ that are conjugate (i.e., there exists a Möbius transformation $P$ so that $M = P \circ N \circ P^{-1}$) have the same number of fixed points, modulo conjugation, the standard forms are

1. \[
\begin{pmatrix}
\cos(\vartheta) & \sin(\vartheta)
\\-\sin(\vartheta) & \cos(\vartheta)
\end{pmatrix}
\] (rotations) in the elliptic case;

2. \[
\begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}
\] (translations) in the parabolic case;

3. \[
\begin{pmatrix}
a & 0 \\
0 & a^{-1}
\end{pmatrix}
\] (dilatations) in the hyperbolic (or loxodromic) case.

### 1.4 Gauss-Bonnet Formula

The first remarkable result in hyperbolic geometry is the Gauss-Bonnet Theorem.

**Theorem 1.4.1** (Gauss-Bonnet). Let $P$ be a hyperbolic triangle (i.e., the sides are geodesics) with interior angles $\alpha$, $\beta$ and $\gamma$. Then

\[
\text{area}_{H^2}(P) = \pi - \alpha - \beta - \gamma.
\]

**Proof.** We divide the proof on two steps:

**Step 1:** We consider the triangle $P$ as in Figure 1.16 (in this case $\gamma = 0$).

![Figure 1.16: Particular geodesic triangle $P$.](image)

We have

\[
\text{area}_{H^2}(P) = \int_{P} \frac{1}{y^2} dx \, dy = \int_{\cos(\beta)}^{\cos(\beta)} \int_{\frac{1}{\sqrt{x^2+y^2}}}^{\infty} \frac{1}{y^2} dy \, dx =
\]
\[ = \int_{\cos(\pi - \alpha)}^{\cos(\beta)} \frac{dx}{\sqrt{x^2 + y^2}} = \int_{\pi - \alpha}^{\beta} \frac{\sin(t)}{\sin(t)} \, dt = \pi - \alpha - \beta \]
as we wanted to prove.

Step 2: Since M"obius transformations preserve the hyperbolic area, we can suppose that the triangle \( P = \Delta(abc) \) has the geodesic \( \overline{ab} \) included in the circle of center 0 and radius 1 and the geodesic \( \overline{ac} \) included in the vertical geodesic \( \overline{a\infty} \), as in Figure 1.17.

![Figure 1.17: Triangle P of Step 2)](image)

We denote with \( \beta' \) the angle \( \hat{c\infty} \).

Using step 1), we have

\[
\text{area}_{\mathbb{H}^2}(P) = \text{area}_{\mathbb{H}^2}(\Delta(ab\infty)) - \text{area}_{\mathbb{H}^2}(\Delta(cb\infty)) = \\
= \pi - \alpha - (\beta + \beta') - [\pi - (\pi - \gamma) - \beta'] = \pi - \alpha - \beta - \gamma
\]

and this concludes the proof of the theorem.

This theorem can be generalized to polygons.

**Theorem 1.4.2.** Let \( P \) be a hyperbolic polygon (i.e., the sides are geodesics) with interior angles \( \alpha_1, \ldots, \alpha_m \). Then

\[
\text{area}_{\mathbb{H}^2}(P) = (n - 2)\pi - \sum_{k=1}^{n} \alpha_k.
\]

Now, we know that a surface of genus \( g \) is made by identifying edges of a \( 4g \)-gon so that all vertices are identified.

Consider a regular geodesic \( 4g \)-gon in the Poincaré disc model \( \Delta \) concentric to the disc (fundamental polygon). The angle sum of a small \( 4g \)-gon is \( \sim (4g - 2)\pi \) which is grater than \( 2\pi \) if the genus \( g > 1 \). For a large \( 4g \)-gon the angle sum is \( \sim 0 \). Since
the angle sum changes continuously, we can find a $4g$-gon with angle sum $2\pi$. As the sides have the same length, we can glue sides by orientation-preserving isometries. In this way, we get a so called hyperbolic atlas for the surface. We use the unique isometries joining the edges to make charts for the edges. As the angle sum is $2\pi$, we get a chart for the vertices. Moreover, applying the isometries to the fundamental $4g$-gon gives a tiling of $\Delta$.

Figure 1.18: Charts of edges and vertices in a genus 2 surface.

1.5 Problems:

1. For $r > 0$ consider $f : [0, 2\pi] \to \mathbb{C}$ given by $f(t) = re^{it}$, which parameterizes the Euclidean circle with Euclidean center 0 and Euclidean radius $r$. By definition, if $\gamma : [a, b] \to \mathbb{C}$ is a path,

$$l_{\mathbb{D}}(\gamma) := \int_{\gamma} \frac{|dz|}{1 - |z|^2}.$$ 

Calculate $l_{\mathbb{D}}(f)$.

2. Given $s > 0$, let $D_s$ be the open hyperbolic disc in $\mathbb{D}$ with hyperbolic center 0 and hyperbolic radius $s$. Show that the hyperbolic area $area_{\mathbb{D}}(D_s)$ of $D_s$ is

$$area_{\mathbb{D}}(D_s) := \int_{D_s} \frac{4}{(1 - |z|^2)^2} dx \, dy = 4\pi \sinh^2\left(\frac{1}{2}s\right).$$

**Hint:** First prove that for $0 < r < 1$

$$d_{\mathbb{D}}(0, r) = \ln\left(\frac{1 + r}{1 - r}\right)$$

and, hence, that $r = \tanh\left[\frac{1}{2}d_{\mathbb{D}}(0, r)\right]$.

From this observation it follows that the hyperbolic radius $s$ of $D_s$ is related to
the Euclidean radius $R$ by $R = \tanh^2(\frac{1}{2}s)$.

Then, using the polar coordinates, calculate

$$\text{area}_{\mathbb{D}}(D_s) = \int_{D_s} \frac{4r}{(1 - r^2)^2} dr d\vartheta.$$  

3. Consider

$$\varphi : \text{PSL}_2(\mathbb{R}) \rightarrow \text{Isom}^+(\mathbb{H}^2)$$

and prove that $\varphi$ is a surjective homomorphism.

**Hint:** It is easy to prove that $\varphi$ is a homomorphism, but it is more difficult to prove that it is surjective. To prove surjectivity, convince yourself that an isometry $\varphi : M \rightarrow N$ between 2-dimensional Riemann surfaces $M$ and $N$ is determined by the image of a point $p$ and the image of a vector in $T_pM$.

So if we prove that we can take the pair $(i, (0, 1))$, where $(0, 1)$ is the vertical unit vector in $T_i\mathbb{H}^2$, in any other pair $(p, v)$ through an element in $\text{PSL}_2(\mathbb{R})$, then we are done.

If $p = x + iy$, you can go first from $i$ to $iy$ and then from $iy$ to $p$.

4. Two elements $M_1$ and $M_2$ in $\text{PSL}_2(\mathbb{R})$ are conjugate if there exists some $P \in \text{PSL}_2(\mathbb{R})$ so that $M_2 = P \circ M_1 \circ P^{-1}$.

Suppose that $M$ and $N$ are elements in $\text{PSL}_2(\mathbb{R})$ conjugate by $P$, so that $M = P \circ N \circ P^{-1}$. Prove that $N$ and $M$ have the same number of fixed points in $\mathbb{H}^2$.

5. Prove that the hyperbolic area in $\mathbb{H}^2$ is invariant under the action of orientation preserving isometries. That is, if $X$ be the set in $\mathbb{H}^2$ whose hyperbolic area $\text{area}_{\mathbb{H}^2}(X)$ is defined and if $A \in \text{PSL}_2(\mathbb{R})$, then

$$\text{area}_{\mathbb{H}^2}(X) = \text{area}_{\mathbb{H}^2}(A(X)).$$

1.6 Solutions:

1. We have

$$l_{\mathbb{D}}(\gamma) = \int_{\gamma} \frac{|dz|}{1 - |z|^2} = \int_0^{2\pi} \frac{|f'(t)|}{1 - |f(t)|^2} dt = \frac{2\pi r}{1 - r^2}.$$  

2. For $0 < r < 1$, we consider $f : [0, r] \rightarrow \mathbb{D}$ given by $f(t) = t$. As the image of $f$ is the hyperbolic segment in $\mathbb{D}$ joining 0 and $r$, we have that, by definition, $d_{\mathbb{D}}(0, r) = l_{\mathbb{D}}(f)$. Hence,

$$d_{\mathbb{D}}(0, r) = l_{\mathbb{D}}(f) = \int_0^r \frac{2}{1 - t^2} dt =$$
1.6. SOLUTIONS:

$$= \int_0^r \left[ \frac{1}{1+t} + \frac{1}{1-t} \right] dt = \ln \left( \frac{1+r}{1-r} \right).$$

Solving for $r$ as a function of $d_\Sigma(0,r)$, we have $r = \tanh \left( \frac{1}{2} d_\Sigma(0,r) \right)$.

Since the hyperbolic radius $s$ of $D_s$ is related to the Euclidean radius $R$ by $R = \tanh \left( \frac{1}{2} s \right)$, then, using the polar coordinates, we have

$$\text{area}_\Sigma(D_s) = \int_{D_s} \frac{4r}{(1-r^2)^2} dr \, d\vartheta = \int_0^R \int_0^{2\pi} \frac{4r}{(1-r^2)^2} dr \, d\vartheta =$$

$$= 2\pi \int_0^R \frac{4r}{(1-r^2)^2} dr = \frac{4\pi R^2}{1 - R^2} =$$

$$= \frac{4\pi \tanh^2 \left( \frac{1}{2} s \right)}{1 - \tanh^2 \left( \frac{1}{2} s \right)} = 4\pi \sinh^2 \left( \frac{1}{2} s \right).$$

3. First we prove that

$$\varphi : \ PSL_2(\mathbb{R}) \rightarrow \text{Isom}^+(\mathbb{H}^2)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (z \mapsto \frac{az+b}{cz+d})$$

is a homomorphism.

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \text{PSL}_2(\mathbb{R}).$$

Then

$$\varphi\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = \varphi\left( \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} \right) =$$

$$= \frac{(aa' + bc')z + (ab' + bd')}{(ca' + dc')z + (cb' + dd')} = a\left( \frac{a'z+b'}{c'z+d'} \right) + b = \frac{a\left( \frac{a'z+b'}{c'z+d'} \right)}{c\left( \frac{a'z+b'}{c'z+d'} \right)} + d =$$

$$= \varphi\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \circ \varphi\left( \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right).$$

Now we want to prove surjectivity. We observe that an isometry $\phi : M \rightarrow N$ between 2-dimensional Riemann surfaces $M$ and $N$ is determined by the image of a point $p$ and the image of a vector in $T_p M$.

So if we prove that we can take the pair $(i, (0,1))$, where $(0,1)$ is the vertical unit vector in $T_i \mathbb{H}^2$, in any other pair $(p,v)$ through an element in $PSL_2(\mathbb{R})$, then we are done.

Let $p = x + iy = x + ie^\lambda$. First we go from $i$ to $x + iy$ through

$$A = \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{array} \right).$$
Notice that $A_+(0,1)$ is a vertical unit vector. So, if we want to get the vector $v$, we need to do a rotation sending $(p, A_+(0,1))$ to $(p, v)$. Hence, the matrix that we are looking for is
\[
\begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos(\vartheta) & \sin(\vartheta) \\
-\sin(\vartheta) & \cos(\vartheta)
\end{pmatrix}
\in PSL_2(\mathbb{R}).
\]

4. If $x$ is a fixed point for $N$, then
\[
M(P(x)) = P \circ N \circ P^{-1}(P(x)) = P(N(x)) = P(x)
\]
and so $P(x)$ is a fixed point for $M$. Moreover, since $N = P^{-1} \circ M \circ P$, if $x$ is a fixed point for $M$, then
\[
N(P^{-1}(x)) = P^{-1} \circ M \circ P(P^{-1}(x)) = P^{-1}(M(x)) = P^{-1}(x)
\]
and so $P^{-1}(x)$ is a fixed point for $N$.

Hence, $M$ and $N$ have the same number of fixed points in $\mathbb{H}^2$.

5. The proof is an application of the change of variables.

Since $a, b, c$ and $d$ are real numbers and $ad - bc = 1$,
\[
A(z) = \frac{az + b}{cz + d} = \frac{(az + b)(cz + d)}{(cz + d)(cz + d)} = \frac{acx^2 + acy^2 + bd + bcx + adx}{(cx + d)^2 + c^2y^2} + i\frac{y}{(cx + d)^2 + c^2y^2}
\]
so we can write
\[
A(x, y) = \frac{acx^2 + acy^2 + bd + bcx + adx}{(cx + d)^2 + c^2y^2}, \quad \frac{y}{(cx + d)^2 + c^2y^2}
\]
The Jacobian is
\[
D(A)(x, y) = \begin{pmatrix}
\frac{(cx + d)^2 - c^2y^2}{((cx + d)^2 + c^2y^2)^2} & \frac{2cy(cx + d)}{((cx + d)^2 + c^2y^2)^2} \\
\frac{-2cy(cx + d)}{((cx + d)^2 + c^2y^2)^2} & \frac{(cx + d)^2 - c^2y^2}{((cx + d)^2 + c^2y^2)^2}
\end{pmatrix}
\]
and the determinant of the Jacobian is
\[
\det(D(A)(x, y)) = \frac{1}{((cx + d)^2 + c^2y^2)^2}.
\]
Then, if we define $m(x, y) = \frac{1}{y^2}$, we have
\[
area_{\mathbb{H}^2}(A(X)) = \int_{A(X)} \frac{1}{y^2} dx dy = \int_X m \circ A(x, y) |\det(D(A))| dx dy =
\]
\[
= \int_X \frac{((cx + d)^2 + c^2y^2)^2}{y^2} \frac{1}{((cx + d)^2 + c^2y^2)^2} dx dy = area_{\mathbb{H}^2}(X).
\]
Chapter 2
Quasi-Conformal Maps

2.1 Dilatation of a linear map

Suppose $D$ is a diagonal 2-dimensional real matrix. Then: $D(x, y) = (ax, by)$ and if $(x, y)$ satisfy $x^2 + y^2 = 1$ then $(ax, by)$ satisfies $\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1$. Hence, a diagonal matrix takes a circle to an ellipse.

Now, let $B$ be an arbitrary matrix. By the singular value decomposition, $B$ can be written as $B = SDR^{-1}$, where $D$ is a diagonal matrix and $R, S$ are rotations - i.e., orthonormal matrices. Therefore, $B$ takes circles to ellipses. Let $t_1, t_2$ be the eigenvalues of $D$.

There are two cases:

- $t_1 = t_2$. Then $D$ is a scalar matrix and $B$ takes circles to circles.
- $t_1 \neq t_2$. In this case, $R(e_1)$ and $R(e_2)$ are taken by $B$ to vectors tangent to the axes of the ellipse. Both $R(e_1) \perp R(e_2)$ and $B(e_1) \perp B(e_2)$.

Let

$$\lambda_{\min} := \min \{ \|B(v)\| \mid \|v\| = 1 \} \quad \lambda_{\max} := \max \{ \|B(v)\| \mid \|v\| = 1 \}.$$ 

Then, by the definitions above, $\{\lambda_{\min}, \lambda_{\max}\} = \{t_1, t_2\}$. Suppose $\lambda_{\min} = t_1$, then $v_{\min} = R(e_1), v_{\max} = R(e_2)$. We will later need:

$$|B| = |SDR^{-1}| = |S||D||R^{-1}| = |D| = \lambda_{\min}\lambda_{\max},$$

where $|A| := \det(A)$.

Definition 2.1.1. If $B$ is a linear map the dilatation of $B$ is $D(B) = \frac{\lambda_{\max}}{\lambda_{\min}}$.

2.2 Regular Quasi-Conformal Mappings

Here we follow the conventions from [8].

Definition 2.2.1. $D \subseteq \mathbb{C}$ is called a Jordan domain if $\pi_1(D) = 1$ and $\partial D$ is a Jordan curve (i.e., $\partial D$ is a simple closed curve).
Definition 2.2.2. Let $w : D \to D'$ a homeomorphism of two Jordan domains $D, D'$ which extends to a homeomorphism on the boundary. Suppose that $w_x, w_y$ exist and are continuous. Denote by $Tw(z)$ the matrix of partial derivatives. The directional derivative at angle $\alpha$ will be denoted $\partial_{\alpha}w(z)$. Let

$$
\lambda_{\text{max}}(z) := \max \{ \partial_{\alpha}w(z) | \alpha \in [0, 2\pi) \},
\lambda_{\text{min}}(z) := \min \{ \partial_{\alpha}w(z) | \alpha \in [0, 2\pi) \}
$$

The dilatation of $w$ at $z$ is:

$$
D_w(z) := D(Tw(z)) = \frac{\lambda_{\text{max}}(z)}{\lambda_{\text{min}}(z)}.
$$

The homeomorphism $w$ is a regular $K$-quasi-conformal map if

$$
\sup_{z \in D} \{ D_w(z) \} \leq K.
$$

We drop the adjective regular if we allow $w$ to have isolated singularities.

**Example 2.2.3.**

0) $f$ is conformal on $D \iff Tw$ doesn’t distort angles $\iff Tw$ takes circles to circles $\iff D_f(z) = 1$ for all $z \iff f$ is 1-quasi-conformal.

1) Consider two rectangles: $R_1$ is the rectangle with vertices at $0, a, a + i, i$ and $R_2$ is the rectangle with vertices at $0, a', a' + i, i$. We assume $a' > a$. Define

Figure 2.1: Every linear transformation can be decomposed as a rotation $R^{-1}$ followed by a dilatation $D$ followed by another rotation $S$. Therefore a circle centered at the origin will map to an ellipse.
2.3. AN OPTIMIZATION PROBLEM

We now compute:

\[ f : R_1 \to R_2 \text{ by } f(x + iy) = \frac{a'}{a}x + iy \text{ we call } f \text{ the affine map from } R_1 \text{ to } R_2. \]

We compute

\[
\begin{pmatrix}
  u_x & v_x \\
  u_y & v_y
\end{pmatrix} = \begin{pmatrix}
  \frac{a'}{a} & 0 \\
  0 & 1
\end{pmatrix}.
\]

Therefore, \( \lambda_{\text{max}} = \frac{a'}{a} \) and \( \lambda_{\text{min}} = 1 \) for all \( z \). So \( f(z) \) is a regular \( \frac{a'}{a} \) quasi-conformal map.

2.3 An optimization problem

**Theorem 2.3.1** (Grötzsch). Let \( w : R_1 \to R_2 \) be any homeomorphism with continuous partial derivatives which extends to a homeomorphism on the boundary and which takes \( (0, a, a + i, i) \) to \( (0, a', a' + i, i) \), then \( w \) is \( K \)-quasi-conformal and \( K \geq \frac{a'}{a} \).

Moreover, if \( K_f = \frac{a'}{a} \) then \( f \) is the affine mapping of \( R_1 \) onto \( R_2 \).

**Proof.** The fact that \( w \) is \( K \)-quasi-conformal follows from the compactness of \( \overline{R_1} \). We now assume that for all \( z \in R_1 \), \( \frac{\lambda_{\text{max}}(z)}{\lambda_{\text{min}}(z)} \leq K \) and prove that \( a' \geq \frac{a^2}{K a} \). We will need the following calculations:

1. \( J_w(z) = |T w(z)| = \lambda_{\text{max}}(z) \lambda_{\text{min}}(z) \geq \frac{1}{K} \lambda_{\text{max}}(z) \geq \frac{1}{K} |w_x(z)|^2 \)

2. by the Chauchy-Schwartz inequality,

\[
\left| \int_0^a 1 \cdot w_x(x + iy) dx \right| \leq \left( \int_0^a 1^2 dx \right)^{\frac{1}{2}} \left( \int_0^a |w_x(x + iy)|^2 dx \right)^{\frac{1}{2}}.
\]

Thus

\[
\int_0^a |w_x(x + iy)|^2 dx \geq \frac{1}{a} \cdot \left| \int_0^a w_x(x + iy) dx \right|^2
\]

With equality if and only if \( w_x(x + iy) \) is a scalar.

3. \( \left| \int_0^a w_x(x + iy) dx \right| = |w(a + iy) - w(iy)| \geq a' \) the last inequality follows since \( w(a + iy) \) is on the line \( x = a \) and \( w(iy) \) is on \( x = 0 \) (here we use the fact that vertices are mapped to vertices).

We now compute:

\[
a' = \text{Area}(R_2) = \int_{R_1} J_w(z) dx dy \geq \int_{R_1} \frac{1}{K} |w_x(x + iy)|^2 dx dy = \\
= \frac{1}{K} \int_0^1 dy \int_0^a |w_x(x + iy)|^2 dx \geq \frac{1}{aK} \int_0^1 dy \left| \int_0^a w_x(x + iy) dx \right|^2 \geq \\
\geq \frac{1}{aK} \int_0^1 dy |a'|^2 = \frac{a'^2}{K a}
\]

The inequality is strict except for when: \( w_x(x + iy) = c \) a scalar, \( \lambda_{\text{max}}(z) = |w_x(z)| \) for all \( z \) and \( \lambda_{\text{max}}(z) = K \) for all \( z \). So \( v_{\text{max}} = x \) hence \( v_{\text{min}} = y \), and \( w_y(x + iy) = \frac{c}{K} \).

Therefore, \( w(x + iy) = cx + i\frac{c}{K} y \). But since vertices are mapped to vertices, \( c = \frac{a'}{a} \) and \( w \) is the affine map from \( R_1 \) to \( R_2 \). \( \square \)
Remark. There is another formula for the dilatation of \(w\). First we define the differential operators \(\partial z, \partial \bar{z}\):

\[
\begin{align*}
  z &= x + iy, & \bar{z} &= x - iy, \\
  x &= \frac{1}{2}(z + \bar{z}), & y &= \frac{1}{2i}(z - \bar{z}).
\end{align*}
\]

Using the chain rule: \(f_z = f_x \frac{\partial z}{\partial x} + f_y \frac{\partial z}{\partial y} = \frac{1}{2}f_x + \frac{1}{2i}f_y = \frac{1}{2}(f_x - if_y)\). Similarly, \(f_{\bar{z}} = \frac{1}{2}(f_x + if_y)\). We have

\[
\begin{align*}
  \frac{\partial}{\partial z} &= \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \\
  \frac{\partial}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\end{align*}
\]

The Beltrami coefficient of \(f(z)\) is

\[
\mu_f(z) = \frac{f_{\bar{z}}(z)}{f_z(z)}.
\]

Notice that if \(f\) is holomorphic and \(f_z \neq 0\) everywhere, then \(\mu_f(z) = 0\).

If \(f\) is orientation preserving, then \(J_w(z) > 0\) for all \(z\). Any easy computation shows that \(J_w(z) = f_z^2 - f_{\bar{z}}^2\). Therefore, \(f_z(z) \neq 0\) and \(\mu_f(z) < 1\).

Fact 2.3.2. \(D_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|} = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}\).

Proof. It is enough to check that \(D_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|} = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}\) for \(f(x + iy) = x + iKy\) since \(f\) is locally of this form. We know that in this case \(D_f = K\). We have:

\[
\begin{align*}
  f_{\bar{z}} &= \frac{1}{2}(1 + i(iK)) = \frac{1}{2}(1 + iK) \\
  f_z &= \frac{1}{2}(1 - i(iK)) = \frac{1}{2}(1 - iK).
\end{align*}
\]

Hence,

\[
\mu_f = \frac{f_{\bar{z}}}{f_z} = \frac{1 - K}{1 + K}
\]

and so

\[
\begin{align*}
  1 + \frac{\frac{1 - K}{1 + K}}{1 - \frac{\frac{1 - K}{1 + K}}{1 + K}} &= 1 + \frac{\frac{K - 1}{1 + K}}{1 - \frac{K - 1}{1 + K}} = K = D_f.
\end{align*}
\]

\(\Box\)

Notice that \(\|D_f\|_\infty \leq K\) if and only if \(\|\mu_f\|_\infty \leq k < 1\). Hence, \(f\) is \(K\)-quasi-conformal if \(\|\mu_f\|_\infty \leq \frac{K - 1}{1 + K}\).

A proof of Grötzsch’s Theorem using this definition can be found in [5].

Corollary 2.3.3. If \(d' > a\) there is no conformal mapping from \(R_1\) to \(R_2\) which takes vertices to vertices.

Proof. If there were a conformal map \(f : R_1 \rightarrow R_2\), Caratheodory’s theorem guarantees that it extends to a homeomorphism of \(\partial R_1\) to \(\partial R_2\). Up to this point there is no contradiction, in fact, the Riemann mapping theorem implies that there is such a map. However, if we further require that vertices are mapped to vertices then by Grötzsch’s theorem \(K_f \geq \frac{d'}{a} > 1\) which is a contradiction. \(\Box\)
We give the definition of absolute continuous function:

**Definition 2.3.4.** A function \( f : [a, b] \to \mathbb{R} \) is absolute continuous if \( E \subset [a, b] \) with \( \mu(E) = 0 \) implies that \( \mu(f(E)) = 0 \).

**Theorem 2.3.5.** If \( f \) is absolute continuous, then \( f' \) exists almost everywhere.

**Definition 2.3.6.** A function \( f : \Omega \to \mathbb{C} = \mathbb{R}^2 \), \( f(x, y) = (f_1(x, y), f_2(x, y)) \) is absolute continuous on lines (ACL) if for all rectangles \( R = [a, b] \times [c, d] \subset \Omega \) the following holds:

1. For almost everywhere \( y \in [c, d] \) the functions \( x \mapsto f_1(x, y) \) and \( x \mapsto f_2(x, y) \) are absolute continuous;
2. For almost everywhere \( x \in [a, b] \) the functions \( y \mapsto f_1(x, y) \) and \( y \mapsto f_2(x, y) \) are absolute continuous.

**Remark.** Notice that (1) implies that there exist partial derivatives of \( f \) respect to \( x \) in almost all the lines \( y = c \) and (2) implies that there exist partial derivatives of \( f \) respect to \( y \) in almost all the lines \( x = c \). If \( f \) is absolute continuous on lines, by Fubini, \( f_x \) and \( f_y \) exist almost everywhere.

**Definition 2.3.7.** A function \( f : \Omega \to \mathbb{C} = \mathbb{R}^2 \) is \( K \)-quasi-conformal if it is a homeomorphism and

1. \( f \) is absolute continuous on lines;
2. the essential supremum \( \|\mu f\|_\infty \leq \frac{K-1}{1+K} \).

**Notation 2.3.8.** We will write quasi-conformal instead of \( K \)-quasi-conformal when it is not important the value of the constant.

A conformal map is 1-quasi-conformal, moreover

**Theorem 2.3.9.** A 1-quasi-conformal map is conformal.

**Example 2.3.10.** \( f : \mathbb{C} \to \mathbb{C}, \ f(x+iy) = x + iKy \) is obviously \( K \)-quasi-conformal.

**Example 2.3.11.** We consider \( f : \mathbb{C} \to \mathbb{C}, \ f(x+iy) = \frac{1}{\sqrt{K}}x + i\sqrt{K}y \) and the following diagram:

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{f} & \mathbb{C} \\
\varphi \downarrow & & \downarrow \varphi \\
\mathbb{C} & \xrightarrow{f} & \mathbb{C}
\end{array}
\]

where \( \varphi(z) = z^2 \). Notice that we can lift \( f \) to \( \tilde{f} \). What does \( \tilde{f} \) look like?

Looking at the diagram we see that the image of the blue curves in Figure 2.2 go to vertical lines through \( \varphi \), then they are expanded via \( f \) and, after the lifting, they come back to the original curves but with the expansion given by \( f \). In a similar way, the image of the red curves in Figure 2.2 go to horizontal lines through \( \varphi \), then they
are contracted via \( f \) and, after the lifting, they come back to the original curves but with the contraction given by \( f \).

\( \tilde{f} \) is not differentiable in 0, but it is differentiable at any point in \( \mathbb{C} \setminus \{0\} \). It is easy to see that \( \tilde{f} \) is absolute continuous on lines and \( \|\mu_f\|_{\infty} \leq \frac{K-1}{1+K} \), so \( \tilde{f} \) is \( K \)-quasi-conformal.

Remark. We can replace \( \varphi(z) = z^2 \) by \( \varphi_n(z) = z^n \) for \( n \in \mathbb{Z}, n \geq 2 \).

Example 2.3.12. The function \( f : \Delta \to \mathbb{C}, f(re^{i\theta}) = \frac{r}{1-r}e^{i\theta} \) is not \( K \)-quasi-conformal for any \( K \). In fact, there are not quasi-conformal maps from \( \Delta \) to \( \mathbb{C} \).

Notice that we already knew that there are not conformal maps from \( \Delta \) to \( \mathbb{C} \) by Liouville’s Theorem.

Example 2.3.13. Let \( C_{\frac{1}{3}} \) be the middle 3rds Cantor’s set and let \( C_{\frac{1}{4}} \) be the Cantor set described in the Figure 2.3.

\[
0 \quad \frac{1}{3} \quad \frac{2}{3} \quad \frac{5}{8} \quad 1
\]

We denote with \( m \) the Lebesgue measure. We have

\[
m(C_{\frac{1}{3}}) = 0, \quad m(C_{\frac{1}{4}}) = 1 - \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n = 1 - \frac{1}{4} \left( \frac{1}{1 - \frac{1}{4}} \right) = \frac{2}{3}.
\]
There is a map $\phi : I \to I$ such that $\phi$ is a homeomorphism and $\phi(C_1) = C_4$. By the previous observation, $\phi$ is not absolute continuous on lines. We can define a measure $\nu$ by $\nu(E) = m(\phi(E) \cap C_1)$. So $\nu(C_{\frac{1}{3}}) \neq 0$ and $\nu$ is a measure on $\mathbb{R}$. Now, we define $\psi(x) = \nu([0, x])$. The graph of $\psi$ is as in Figure 2.4.

Let $\Psi(x, y) = (x, \psi(x) + y) = x + i(\psi(x) + y)$. In particular, $\Psi_z = 0$ almost everywhere since $\Psi_x = 1 + i\psi'(x)$ (when $\psi'(x)$ is defined) and $\Psi_y = i$, but $\Psi$ is not absolute continuous on lines and so it is not quasi-conformal.

**Facts about quasi-conformal maps:**

We will explain why the quasi-conformal maps are important. The proof of the Theorems 2.3.15 and 2.3.16 can be found in [5].

**Definition 2.3.14.** The total derivative of $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ at $x \in \Omega$ is a linear function $Df_x : \mathbb{R}^2 \to \mathbb{R}^2$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $|h| < \delta,$ then

$$\left| \frac{f(x + h) - f(x) - Df_x(h)}{|h|} \right| < \varepsilon.$$  

**Theorem 2.3.15.** If $f$ is quasi-conformal, then $f$ has a total derivative almost everywhere.

**Theorem 2.3.16.** If $f : \Omega \to \mathbb{C}$ is quasi-conformal, then for all $\phi \in C^\infty_0(\Omega)$, where $C^\infty_0(\Omega)$ denotes the set of $C^\infty(\Omega)$-functions with compact support in $\Omega$,

$$\int \int_{\Omega} f_z \cdot \phi = -\int \int_{\Omega} f \cdot \phi_z, \quad (2.1)$$

$$\int \int_{\Omega} f_{\bar{z}} \cdot \phi = -\int \int_{\Omega} f \cdot \phi_{\bar{z}} \quad (2.2)$$

and $f_z$ and $f_{\bar{z}}$ are locally in $L^2$. 

---

**Figure 2.4:** The graph of $\psi$. 

---
Definition 2.3.17. If $f : \Omega \to \mathbb{C}$ has $f_z$ and $f_{\bar{z}}$ such that 2.1 and 2.2 hold, then $f_z$ and $f_{\bar{z}}$ are called *distributional derivatives*.

We can give another definition of quasi-conformal maps:

**Definition 2.3.18.** The map $f : \Omega \to \mathbb{C}$ is quasi-conformal if it is a homeomorphism and

1. $f$ has distributional derivatives $f_z$ and $f_{\bar{z}}$ locally in $L^2$;
2. $\|f_z\|_\infty \leq \frac{K-1}{1+K}$. 

**Theorem 2.3.19** (Compactness Theorem). Let $\Omega \subset \hat{\mathbb{C}}$ be open and connected, $a_1, a_2, a_3 \in \Omega$ and let $A_1, A_2, A_3 \subset \hat{\mathbb{C}}$ be disjoint compact sets. Define

$$\mathfrak{F}(K; A_1, A_2, A_3) := \{ f : \Omega \to \mathbb{C} \mid f \text{ is } K\text{-quasi-conformal and } f(a_i) \in A_i, i = 1, 2, 3 \}.$$  

Then $\mathfrak{F}$ is compact in the compact-open topology (i.e., if $\{f_j\} \in \mathfrak{F}$, then there exists a subsequence $\{f_{n_j}\}$ such that for every compact $B \subset \Omega$, $\{f_{n_j}\}$ converges uniformly on $B$).

We need the Arzelà-Ascoli Theorem, but in order to use the Arzelà-Ascoli Theorem we need another definition of quasi-conformality. Before we give the definition of quasi-conformal maps, we must introduce the module of a quadrilateral.

### 2.3.1 The Module of a Quadrilateral

**Definition 2.3.20.** A *quadrilateral* $(Q, q_1, q_2, q_3, q_4)$ is a closed Jordan domain $Q$ with four distinguished points.

**Proposition 2.3.21.** For every quadrilateral $(Q, q_1, q_2, q_3, q_4)$ there is a rectangle $R = [0, a] \times [0, b]$ and a homeomorphism $h : Q \to R$ which extends to a homeomorphism on the boundaries such that $h$ is conformal in $\text{int}(Q)$ and $h(q_1) = 0, h(q_2) = a, h(q_3) = a + ib, h(q_4) = b$.

Moreover, $\frac{\partial}{\partial \bar{z}}$ is independent of $h$. It is denoted $M(Q, q_1, q_2, q_3, q_4)$ or $m(Q)$ and called the *modulus of the quadrilateral* $(Q, q_1, q_2, q_3, q_4)$.

**Example 2.3.22.** Let $Q = \mathbb{H}^2$ the Upper Half Plane and the four points are $-x, -1, 1, x$ where $1 < x \in \mathbb{R}$. Let $R_a = [0, a] \times [0, 1]$ be a rectangle. One way to see that there exists a conformal map $h : Q \to R_a$ for some $a$ is the following. Map $Q$ to the unit disc $U$ and $-x, -1, 1$ to $1, i, -1$ using the Riemann mapping theorem. $x$ will be mapped to some point $p$ on the arc $(-1, 1)$ determined by the cross-ratio$^1$ $[-x : -1 : 1 : x]$. Similarly, map $R_a$ onto $U$ and $0, a, i + a$ to $1, i, -1$. $i$ will be mapped to $q$ some point on the arc $(-1, 1)$. As we vary $a$, the cross-ratio $[0 : a : a + i : i]$ varies continuously, and $q$ moves continuously along the arc. There exists a rectangle $R_a$ with the same cross ratio as $(\mathbb{H}^2, -x, -1, 1, x)$ and the construction described above will yield a map

---

$^1$Here we must believe that the cross-ratio of four points is invariant under conformal maps.
with the right properties.
Surprisingly, in this case we can actually write down the map:

\[ f(z) = \int_0^z \frac{dw}{\sqrt{(w+x)(w+1)(w-1)(w-x)}} \]

is a conformal map from \( \mathbb{H}^2 \) onto some rectangle \([-K, K] \times [0, K']\). For details see [9].

**Proof of Proposition 2.3.21.** By the Riemann mapping theorem there’s a conformal map which takes \((Q, q_1, q_2, q_3, q_4)\) onto \((\mathbb{H}^2, r_1, r_2, r_3, r_4)\). We claim that there is a Möbius transformation \(\phi: \mathbb{H}^2 \to \mathbb{H}^2\) taking \(r_1, r_2, r_3, r_4\) to \(-x, -1, 1, x\). The geodesics \([r_1, r_3]\) and \([r_2, r_4]\) intersect at \(z_0\) with angle \(\alpha\). Let \(z_1(x)\) be the intersection point of \([-x, 1], [-1, x]\). If \(x\) is close to 1 the angle at the intersection is close to \(\pi\), if \(x\) is close to \(\infty\) then the angle at the intersection point is close to 0. Thus there is one \(x\) for which the angle at the intersection point is \(\alpha\). Let \(\phi\) be the element of \(PSL_2(\mathbb{R})\) determined by taking \(z_0\) to \(z_1(x)\) and the tangent vector to \([r_1, r_3]\) at \(z_0\) to the tangent vector to \([-x, 1]\) at \(z_1(x)\). \(\phi\) takes \(r_1, r_2, r_3, r_4\) to \(-x, -1, 1, x\).

Now, we use the map in Example 2.3.22 to map it onto a rectangle. Since all of the maps above are conformal, so is their composition, and by Caratheodory, it extends to a homeomorphism on the boundary.
As for the uniqueness, it follows from Grötzsch’s theorem (or by the reflection principle). \(\square\)

**Definition 2.3.23.** We say that \(f: \Omega \to \mathbb{C}\) is \(K\)-quasi-conformal if it is a homeomorphism onto its image and for all quadrilaterals \(Q\)

\[ \frac{1}{K} m(Q) \leq m(f(Q)) \leq K m((Q)) \]

This definition is equivalent to the previous one.

### 2.3.2 The Module of an Annulus

An *annulus* is just a topological annulus \(A\), that is, it is a surface homeomorphic to \(S^1 \times (0, 1)\).

**Proposition 2.3.24.** There exists a conformal map \(f: A \to A(r, R)\), where \(A(r, R)\) is the standard annulus (see Figure 2.5).

Notice that we can rescale the radii of the annulus \(A(r, R)\) through a conformal map so that the radius \(r\) is equal to 1 and we call this new annulus \(A_0\). Up to scaling, the standard annulus \(A_0\) associated to \(A\) is unique.

**Proof.** We consider the universal cover \(\tilde{A}\) of \(A\) and \(\tilde{A}_0\) of \(A_0\) and we use the Uniformization Theorem to go from \(\tilde{A}\) to a disc and from the disc to \(\tilde{A}_0\) (see Figure 2.6). \(\square\)
Figure 2.5: The standard annulus $A(r, R)$.

Figure 2.6: Proof of the Proposition 2.3.24.

**Definition 2.3.25.** We define the modulus of the annulus $A$ as

$$m(A) := \frac{1}{2\pi} \log\left(\frac{R}{r}\right),$$

where $r$ and $R$ are the radii of the standard annulus $A(r, R)$ associated to $A$.

Since the covering map $\pi : R \to A$ is the exponential map, $m(R) = m(A)$. The *modulus of an annulus* is useful because if $f : X \to Y$ is a $K$-quasi-conformal map between two surfaces, then there exists a relationship between the length of a curve $\gamma$ in $X$ and the associate annulus $A_X$ in the covering space:

$$l_{\gamma}(X) = \frac{\pi}{m(A_X)}.$$

If we can control how $m(A_X)$ changes, then we can control the length of the curve (see Figure 2.7).
Definition 2.3.26. We say that \( f : \Omega \to \mathbb{C} \) is \( K\)-quasi-conformal if it is a homeomorphism onto its image and for all annulus \( A \)

\[
\frac{1}{K} m(A) \leq m(f(A)) \leq Km((A)).
\]

2.4 Extremal Length

Definition 2.4.1. Let \( \Omega \subset \mathbb{C} \) be a region and \( \rho : \Omega \to [0, \infty) \) be a function (not necessarily continuous or smooth). For every \( v \in T_z\Omega \), then \( |v|_\rho = \rho(z)|v| \) is a conformal metric, where \( |v| \) is the Euclidean length.

If \( \gamma(t) \) with \( a \leq t \leq b \) is a piecewise smooth path, then

\[
|\gamma|_\rho = \int_a^b |\gamma'(t)|\rho dt = \int_a^b \rho(\gamma(t))|\gamma'(t)|dt.
\]

Similarly,

\[
\text{area}(\rho) = \int_{\Omega} \rho^2 dx dy.
\]

Let \( Q(z_1, z_2, z_3, z_4) \) be a quadrilateral and \( \Gamma \) be the set of piecewise smooth paths in \( Q \) from the side \( z_1z_4 \) to \( z_2z_3 \).

Let \( \rho \) be a conformal metric on \( Q \), then

\[
L_\Gamma(\rho) := \inf_{\gamma \in \Gamma} |\gamma|_\rho.
\]

In order to get extremal length we need to normalize.

Definition 2.4.2. The extremal length \( E(Q, \Gamma) \) is given by

\[
\sup_{\rho} \frac{L_\Gamma(\rho)^2}{\text{area}(\rho)}.
\]
Lemma 2.4.3. If \( \lambda \in (0, \infty) \), then
\[
\frac{L_{\Gamma}(\rho)^2}{\text{area}(\rho)} = \frac{L_{\Gamma}(\lambda \rho)^2}{\text{area}(\lambda \rho)}
\]

The proof follows trivially from the definition. We would like to see that \( E(Q, \Gamma) = m(Q) \).

Lemma 2.4.4. Let \( Q, Q' \) be quadrilaterals and \( f : Q \to Q' \) be a conformal map taking marked points to marked points. Then \( E(Q, \Gamma) = E(Q', \Gamma') \).

Proof. Let \( \rho' \) be a conformal metric on \( Q' \). Define \( \rho \) on \( Q \) to be
\[
\rho(x) := |f'(x)|\rho'(f(x)).
\]
Then since \( \Gamma' = f(\Gamma) \) we see that
\[
\frac{L_{\Gamma}(\rho)^2}{\text{area}(\rho)} = \frac{L_{\Gamma'}(\rho')^2}{\text{area}(\rho')}
\]
and hence \( E(Q, \Gamma) = \sup_{\rho} \frac{L_{\Gamma}(\rho)^2}{\text{area}(\rho)} = \sup_{\rho'} \frac{L_{\Gamma'}(\rho')^2}{\text{area}(\rho')} = E(Q', \Gamma') \).

Lemma 2.4.5. If \( R \) is the rectangle in \( \mathbb{C} \) with vertices \( 0, a, a+ib, ib \), then \( E(R, \Gamma) = \frac{a}{b} \).

Proof. Let \( \rho \) be a conformal metric on \( R \). Then by Cauchy-Schwartz we have
\[
\text{Area}(\rho) \cdot ab = \int_R \rho^2 \int_R 1^2 \geq \left( \int_0^b \int_0^a \rho \cdot 1 \, dx \, dy \right)^2 \geq \left( \int_0^b L_{\Gamma}(\rho) \, dy \right)^2 = (b \cdot L_{\Gamma}(\rho))^2.
\]
Thus
\[
\frac{L_{\Gamma}(\rho)^2}{\text{area}(\rho)} \leq \frac{a}{b}.
\]
This implies that \( E(R, \Gamma) \leq \frac{a}{b} \). If \( \rho \) is the constant function 1, then
\[
\frac{L_{\Gamma}(\rho)^2}{\text{area}(\rho)} = \frac{a}{b}
\]
and hence \( E(R, \Gamma) = \frac{a}{b} \).
Corollary 2.4.6. \( m(Q) = E(Q, \Gamma) \).

Proof. Extremal length is invariant under conformal maps. By Grötzsch’s Theorem, we can end \( Q \) conformally to the rectangle in \( \mathbb{C} \) with vertices \( 0, a, a + ib, ib \). Then applying Lemma 2.4.5, \( m(Q) = \frac{a}{b} = E(R, \Gamma) \).

Let \( Q^t(z_1, z_2, z_3, z_4) = Q(z_2, z_3, z_4, z_1) \). Let \( \Gamma^t \) denote the paths for \( Q^t \). Then notice that \( m(Q^t) = \frac{1}{m(Q)} \).

\[
m(Q^t) = E(Q^t, \Gamma^t) = \sup_{\rho} \frac{L_{\Gamma^t}(\rho)^2}{\text{area}(\rho)} = \frac{1}{m(Q)},
\]
so \( m(Q) = \inf_{\rho} \frac{\text{area}(\rho)}{L_{\Gamma^t}(\rho)^2} \). If \( \rho \) is a conformal metric on \( Q \), then

\[
\frac{L_{\Gamma}(\rho)^2}{\text{area}(\rho)} \leq m(Q) \leq \frac{\text{area}(\rho)}{L_{\Gamma^t}(\rho)}.\]

If \( \rho = 1 \), the above inequality is called Rengel’s inequality.

Example 2.4.7. Let \( Q \) be the whole quadrilateral and \( Q_0 \) the subquadrilateral as in Figure 2.9.

\[
Q
\]

\[
Q_0
\]

Figure 2.9: The subquadrilateral \( Q_0 \).

Let \( \rho_0 \) be a conformal metric on \( Q_0 \) and \( \rho \) the conformal metric given by

\[
\rho(x) = \begin{cases} 
\rho_0(x) & x \in Q_0 \\
0 & x \in Q - Q_0.
\end{cases}
\]

Then

\[
m(Q_0) = \frac{\text{area}(\rho_0)}{L_{\Gamma^t}(\rho_0)^2} = \frac{\text{area}(\rho)}{L_{\Gamma^t}(\rho)} \geq m(Q).
\]

Definition 2.4.8. (Extremal length for annuli) In this case, \( \Gamma \) will be the set of paths from one boundary component to the other boundary component. Then

\[
E(A, \Gamma) = \sup_{\rho} \frac{L_{\Gamma}(\rho)^2}{\text{area}(\rho)}.
\]

Let \( A(r, R) := \{(t, \theta) \in \mathbb{R}^2 \mid r \leq t \leq R, \theta \in [0, 2\pi)\} \).
Lemma 2.4.9. Let $A = A(r, R)$. Then $E(A, \Gamma) = \frac{1}{2\pi} \log \left( \frac{R}{r} \right)$.

Proof. Apply the same argument to some fundamental domain of the universal covering of the annulus which is a strip. \hfill \Box

The nice metric on the annulus is $\rho(z) = \frac{1}{|z|}$. This metric arises by taking the natural map from the bi-infinite Euclidean cylinder and mapping it to $\mathbb{C}$. Under this conformal map the standard Euclidean metric pushes forward to $\frac{1}{|z|}$.

In this case, $\Gamma^t := \{\text{simple closed paths separating the boundary components}\}$.

Let $E(A, \Gamma^t) = \sup_{\rho} \frac{L_{\Gamma^t}(\rho)^2}{\text{area}(\rho)}$.

Lemma 2.4.10. If $A = A(r, R)$, then

$$E(A, \Gamma^t) = \frac{2\pi}{\log \left( \frac{R}{r} \right)} = \frac{1}{m(A)}.$$ 

Proof. Apply Cauchy-Schwartz to $\rho$ and $\frac{1}{|z|}$. \hfill \Box

Define $m(A) = \sup_{\rho} \frac{L_{\Gamma^t}(\rho)^2}{\text{area}(\rho)} = \inf_{\rho} \frac{\text{area}(\rho)}{L_{\Gamma^t}(\rho)^2}$.

Example 2.4.11. Let $A$ be an annulus with $\text{diam(inner boundary)} \geq \epsilon$ and $\text{diam}(A) \leq R$.

If $\rho = 1$, then $\text{Area}(\rho) \leq \pi \left( \frac{R}{2} \right)^2$ and $L_{\Gamma^t}(\rho) \geq 2\epsilon$. This implies that

$$m(A) \leq \frac{\pi R^2}{(2\epsilon)^2}.$$ 

Theorem 2.4.12. Let $\Delta = \{z \in \mathbb{C} \mid 0 < |z| \leq 1\}$. If $f : \Delta \to \mathbb{C}$ is quasi-conformal and bounded then $f$ extends continuously to 0.
Proof. Let $A = \{z \in \mathbb{C} | 0 < |z| \leq 1\}$, then $area(A) = \infty$. Let $C_r = \{z \in \mathbb{C} ||z| = r\}$. $f$ extends continuously if and only if for all $\epsilon$ there exists a $\delta$ such that if $r < \delta$ then $diam(f(C_r)) < \epsilon$. For the standard annulus $A(r, R)$ we see that

$$m\left(A\left(r, \frac{1}{2}\right)\right) = \frac{1}{2\pi}\log\left(\frac{1}{2r}\right) \to \infty \text{ as } r \to \infty.$$ 

But if $f$ does not extend then $m(f(A(r, \frac{1}{2})))$ is uniformly bounded above. Since $m(f(A(r, \frac{1}{2}))) \geq \frac{1}{R}m(A(r, \frac{1}{2}))$, $m(A(r, \frac{1}{2}))$ is uniformly bounded and this gives a contradiction. \hfill \Box

Corollary 2.4.13. There does not exist a quasi-conformal map from $\mathbb{C}$ to $\Delta$.

Proof. Assume that $f : \mathbb{C} \to \Delta$ is such a map. Let $g(z) = f(\frac{1}{z})$ then $g$ is defined on $\Delta^*$ and is bounded. This implies that $g$ extends to zero and hence $f$ extends to infinity. This is a contradiction since $\hat{\mathbb{C}}$ and $\Delta$ are not homeomorphic. \hfill \Box

2.4.1 Proof of the Compactness Theorem

We gave three definitions of quasi-conformal map: the analytic definition, the quadrilateral definition and the annulus definition. By Grötzsch’s Theorem, the analytic definition implies the quadrilateral definition. Moreover, if we cut an annulus $A$ along two arcs, then we get two quadrilaterals $Q_1$ and $Q_2$. Comparing $m(A)$ with $m(Q_1) + m(Q_2)$ it is easy to see that we get the equality if $A = A(r, R)$. So, using the standard annuli, we see that the quadrilateral definition implies the annulus definition.

We will give a sketch of the proof of the Compactness Theorem 2.3.19 using the quadrilateral definition and the fact that the quadrilateral definition implies the annulus definition.

Proof. We will divide the proof in many steps.

Step (0): We reduce to the case where $A_i = \{a_i\}$.

If $f \in \mathcal{F}$, then there exists a unique Möbius transformation $\phi$ such that $\phi(f(a_i)) = a_i$ and

$$\{\phi \in \text{Möb} = PSL_2(\mathbb{C}) | \phi^{-1}(a_i) \in A_i\}$$

is compact.

Let assume that we proved Step (0).

We will use the spherical metric on $\hat{\mathbb{C}}$ which is the metric conformal to the standard metric on the sphere.

Theorem 2.4.14 (Arzela-Ascoli). Let $X$ and $Y$ be two metric spaces and $f_n : X \to Y$ be continuous maps such that

1. $f_n$ are equicontinuous i.e., $\forall \epsilon > 0 \exists \delta > 0$ such that if $\forall x_0, x_1 \in X$ so that $d(x_0, x_1) < \delta$, then $d(f_n(x_0), f_n(x_1)) < \epsilon$ for all $n$;

2. $\{f_n(x)\}$ is compact for all $x \in X$. 
Then $f_n$ has a uniformly convergent subsequence.

**Step (1):** Let $f_n \in \mathcal{F}$. Then $f_n$ has a subsequence that converges uniformly on compact sets.

We know that there exist countable compact sets $C_1 \subset C_2 \subset \cdots$ with $\Omega = \bigcup C_i$.

Let assume that we have Step (1) for a compact set. Then inductively we construct subsequences $\{f_{i,j}\}$ such that $\{f_{i,j}\}$ converges uniformly on $C_i$ and $\{f_{i+1,j}\}$ is a subsequence of $\{f_{i,j}\}$. By the diagonal argument, $\{f_{i,i}\}$ converges uniformly on all $C_j$.

Since any compact set is contained in some $C_j$, $\{f_{i,i}\}$ converges uniformly on all compact sets.

It remains to prove:

**Step (1‘):** Let $f_n \in \mathcal{F}$ and $C \subset \Omega$ be a compact set. Then $f_n$ has a subsequence that converges uniformly on $C$.

We want to apply the Arzela-Ascoli Theorem.

- **Equicontinuity:** Fix $\varepsilon > 0$. Choose $r > 0$ such that $\varepsilon < r$ and

  1. $d(a_i, a_j) > 2r$ for $i, j \in \{1, 2, 3\}$;
  2. If $z \in C$, $B(z, r) \subset \Omega$.

Then for every $z \in C$, at most one of $a_1, a_2, a_3$ is in $B(z, r)$. We can assume that $a_2, a_3 \notin B(z, r)$. We assume that the ball $B(z, r)$ is so small that the euclidean metric is equivalent to the spherical metric.

Let $A := A(z, \delta, r)$ denote the annulus with center $z$, inner radius $\delta$ and outer radius $r$.

Choose $\delta > 0$ such that $m(A(z, \delta, r)) > \frac{K\pi r^2}{\varepsilon^2}$.

Let $z' \in C$ such that $d(z, z') < \delta$.

Let $f \in \mathcal{F}$. Since $f$ is $K$-quasi-conformal, we have

$$m(f(A(z, \delta, r))) \geq \frac{1}{K} m(A(z, \delta, r)) > \frac{\pi r^2}{\varepsilon^2}.$$ 

Let $\eta := \min\{d(f(z), f(z')), d(a_2 = f(a_2), a_3 = f(a_3))\}$. Notice that $d(a_2 = f(a_2), a_3 = f(a_3)) > 2r$.

We want an upper bound of $m(f(A(z, \delta, r)))$ in terms of $\eta$:

$$m(f(A(z, \delta, r))) = \inf_{\rho} \frac{\text{area}(\rho)}{L^{\ast}(\rho)^2} \leq \frac{4\pi r^2}{(2\eta)^2} = \frac{\pi r^2}{\eta^2}$$

and so

$$\frac{\pi r^2}{\eta^2} > \frac{\pi r^2}{\varepsilon^2}.$$ 

Hence, $\eta < \varepsilon < r$.

If $\eta < 2r$, then $\eta = d(f(z), f(z'))$ and so $d(f(z), f(z')) < \varepsilon$.

- $\{f_n(x)\}$ is compact for every $x$ since the sphere $\hat{\mathbb{C}}$ is compact.
By the Arzela-Ascoli Theorem, there exists a subsequence \( \{f_n\} \) which converges uniformly to \( f \) and \( f \) is continuous.

**Step (2):** Let \( f_n \in \mathcal{F} \) and \( f_n \to f \) uniformly on compact sets. Then, at each \( z \in \Omega \), \( f \) is either locally injective or locally constant.

By contradiction, we fix \( B(z,r) \subset \Omega \) and we assume that there exist \( z_1, z_2, z_3 \in B(z,r) \) such that \( f(z_1) \neq f(z_2) = f(z_3) \). We connect these points with spherical arcs as in Figure 2.12.

Since \( f(z_1) \neq f(z_2) \), there exists \( r \) such that \( d(f_n(z_1), f_n(z_2)) > r \) for every \( n \).

Moreover, \( \varepsilon_n := d(f_n(z_2), f_n(z_3)) \to 0 \), because \( f(z_2) = f(z_3) \).

Assume \( 2\varepsilon_n < r \). Let \( A_n := A(f_n(z_2), 2\varepsilon_n, r) \).

Both the boundary components of \( A_n \) intersect both the boundary components of \( f_n(A) \). Hence, each boundary components intersect the annulus \( f_n(A) \) twice. Define

\[
\rho_n(z) := \begin{cases} 
\frac{1}{|z - f_n(z)|}, & \text{if } z \in f_n(A) \cap A_n \\
0, & \text{otherwise}
\end{cases}
\]

Since \( L_\Gamma(\rho_n) \geq 2 \log\left(\frac{r}{2\varepsilon_n}\right) \), we have

\[
m(f_n(A)) \leq \frac{\text{area}(\rho_n)}{L_\Gamma(\rho_n)^2} \leq \frac{2\pi \log\left(\frac{r}{2\varepsilon_n}\right)}{(2 \log(\frac{r}{2\varepsilon_n}))^2} \to 0,
\]

where \( L_\Gamma(\rho_n) \) is the set of closed curves that separate the boundary components.

We proved that any sequence \( f_n \in \mathcal{F} \) has a subsequence that converges uniformly on compact sets.

We need to show: if \( f_n \in \mathcal{F} \), \( f_n \to f \), then \( f \in \mathcal{F} \).

**Step (3): ** \( f \) is injective.

We know that at every point \( f \) is either locally constant or locally injective. Obviously, \( \text{locally constant} \) is an open condition. If \( z_i \to z \) and \( f \) is locally constant at \( z_i \), then \( f \) is not locally injective at \( z \) because \( \text{locally injective} \) is an open condition and so \( f \) is locally constant at \( z \). Therefore, \( \text{locally constant} \) is a closed condition. Since \( \Omega \) is connected, if there is one point where \( f \) is locally constant, then \( f \) is constant.

Since \( f \) is not constant because \( f(a_i) = a_i \) for \( i = 1, 2, 3 \), \( f \) is locally injective.

Now, let \( z \in \Omega \) and \( B = B(z,r) \subset \Omega \) such that \( f \) is injective on \( B \).

Since \( f \) is locally injective we have the following fact:
CHAPTER 2. QUASI-CONFORMAL MAPS

Fact 2.4.15. There exists $\varepsilon > 0$ such that $f_n(B) \supset B(f(z), \varepsilon)$ for every $n$ sufficiently large.

Since $f_n$ is globally injective, if $z' \in \Omega \setminus B$, then $f_n(z') \notin B(f(z), \varepsilon)$. Therefore, $f(z') \neq f(z)$. Hence $f$ is injective and $f(a_i) = a_i$, for $i = 1, 2, 3$, since $f_n(a_i) = a_i$, for $i = 1, 2, 3$ for every $n$.

The last thing to show is the $K$-quasi-conformality of $f$.

**Step (4): $f$ is $K$-quasi-conformal.**

We will show that if $Q$ is a quadrilateral, then $m(f_n(Q)) \to m(f(Q))$ as $n \to \infty$.

Let $R$ a rectangle and $Q$ a quadrilateral with vertices in the squares as in Figure 2.13.

We put the standard euclidean metric on $R$.

Fact 2.4.16. Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $\omega < \delta$, then $|m(Q) - m(R)| < \varepsilon$.

Let $Q \subset \Omega$.

Fact 2.4.17. For large $n$ there exists $Q_n \subset Q$ such that $f_n(Q_n) \subset f(Q)$, $m(Q_n) \to m(Q)$ as $n \to \infty$ and $f_n(Q_n)$ is in a $\delta$-neighborhood of $f(Q)$.

So, taking the limit as $n \to \infty$,

$$\frac{1}{K} m(Q_n) \leq m(f_n(Q_n)) \leq K m(Q_n)$$

$$\frac{1}{K} m(Q) \leq m(f(Q)) \leq K m(Q).$$

This concludes the proof of the theorem. \qed
2.5 Grötzsch’s Theorem in the general form

We proved Grötzsch’s Theorem in the case of differentiable functions. The aim of this section is to give the generalization of Grötzsch’s Theorem in the case of almost everywhere differentiable functions and one application of this theorem.

**Lemma 2.5.1.** Let \( Q = Q(z_1, z_2, z_3, z_4), Q_1 = Q(z_1, z_5, z_6, z_4) \) and \( Q_2 = Q(z_5, z_2, z_3, z_6) \) such that \( Q = Q_1 \cup Q_2, Q_1 \cap Q_2 = \emptyset \). Then \( m(Q) \geq m(Q_1) + m(Q_2) \). We get the equality if and only if \( Q_1 \) and \( Q_2 \) are rectangles.

**Proof.** Let \( \rho = 1 \) on \( Q \) and \( \rho_1 = \rho|Q_1, \rho_2 = \rho|Q_2 \). By the definitions of \( \rho, \rho_1 \) and \( \rho_2 \),

\[
L_{\Gamma_1}(\rho_1) \geq L_{\Gamma_2}(\rho), \quad L_{\Gamma_2}(\rho_2) \geq L_{\Gamma_2}(\rho).
\]

Therefore,

\[
m(Q_1) + m(Q_2) \leq \frac{\text{area}(\rho_1)}{L_{\Gamma_1}(\rho_1)^2} + \frac{\text{area}(\rho_2)}{L_{\Gamma_2}(\rho_2)^2} \leq \frac{\text{area}(\rho)}{L_{\Gamma}(\rho)^2} \leq m(Q).
\]

Hence, \( m(Q) \geq m(Q_1) + m(Q_2) \).

The last part of the statement is trivial. \( \square \)

We now can prove Grötzsch’s Theorem in the general form:

**Theorem 2.5.2** (Grötzsch). Let \( R_1 \) and \( R_2 \) be rectangles in \( \mathbb{C} \) with vertices \((0, a, a + i, i)\) and \((0, a', a' + i, i)\), \( a' > a \). If \( f \) is \( K \)-quasi-conformal, then \( K \geq \frac{a'}{a} \). Moreover, \( K = \frac{a'}{a} \) if and only if \( f \) is the affine mapping of \( R_1 \) onto \( R_2 \).

**Proof.** We consider two rectangles \( Q_1 \) and \( Q_2 \) in \( R' \) as in Figure 2.15.

![Figure 2.15: The rectangles \( Q_1 \) and \( Q_2 \) in \( R' \).](image)

By Lemma 2.5.1 and the fact that \( f \) is \( K \)-quasi-conformal,

\[
\frac{1}{K}a' = \frac{1}{K}(m(Q_1) + m(Q_2)) \leq m(f^{-1}(Q_1)) + m(f^{-1}(Q_2)) \leq m(R) = a
\]

and so \( K \geq \frac{a'}{a} \).

We have the equality \( K = \frac{a'}{a} \) if and only if \( f^{-1}(Q_1) \) and \( f^{-1}(Q_2) \) are obtained from \( Q_1 \) and \( Q_2 \) scaling by \( K \).
So \( f^{-1}(Q_1) \) and \( f^{-1}(Q_2) \) are rectangles if and only if there exists \( \tilde{x} \) such that \( f^{-1}(x + iy) = \tilde{x} + i\tilde{y} \). We have

\[
m(f^{-1}(Q_1)) = \tilde{x}, \quad m(f^{-1}(Q_2)) = a - \tilde{x}
\]

and

\[
\frac{x}{K} \leq \tilde{x} \leq Kx, \quad \frac{(a' - x)}{K} \leq (a - \tilde{x}) \leq K(a' - x).
\]

Then

\[
\frac{1}{K} \leq \frac{\tilde{x}}{x} \leq K, \quad \frac{1}{K} \leq \frac{a - \tilde{x}}{a' - x} \leq K.
\]

If \( \frac{\tilde{x}}{x} > \frac{1}{K} \), then \( \frac{1}{K} > \frac{a - \tilde{x}}{a' - x} \), which is a contradiction. So \( \frac{\tilde{x}}{x} = \frac{1}{K} \), that is \( \tilde{x} = \frac{x}{K} \).

Similarly, for the \( y \)-coordinate we use \( f \) and we draw a horizontal line to define the rectangles \( Q_1 \) and \( Q_2 \) in \( R \). The calculation is the same as for the \( x \)-coordinate.

Using the geometric definition of the modulus and the Theorem 2.5.2 it is easy to prove the following

**Theorem 2.5.3.** If \( f \) is 1-quasi-conformal, then \( f \) is conformal.

It is possible to prove the theorem using the analytic definition. The problem is that we don’t know that the composition \( g \circ f \) of a quasi-conformal map \( f \) and a conformal map \( g \) is quasi-conformal. The difficult part to show is the ACL condition. So the proof uses the geometric definition.

**Proof.** We consider a square \( Q \). So \( f(Q) \) is a quadrilateral. Since \( f \) is 1-quasi-conformal, there exists a map \( g \) which sends the quadrilateral \( f(Q) \) to a square \( Q' \). Since \( g \circ f \) sends the square \( Q \) to the square \( Q' \), by Grötzsch’s Theorem in the general form, \( g \circ f \) is the affine map. Hence, \( f \) is conformal. \( \square \)
Chapter 3

Teichmüller Space and Moduli Space

3.1 Riemann Surfaces

Let $S$ be a topological surface, i.e., a Hausdorff paracompact 2-manifold. A chart on $S$ is a pair $(U, \phi)$ where $U \subseteq S$ is open and $\phi : U \to \mathbb{C}$ is a homeomorphism onto its image. An atlas $\mathcal{A}$ is a collection of charts such that $\forall x \in S$ there exists $(U, \phi) \in \mathcal{A}$ with $x \in U$.

Let $(U_0, \phi_0), (U_1, \phi_1)$ be charts, where $U_0 \cap U_1$ is non-empty. The map $\phi_1 \circ \phi_0^{-1} : \phi_0(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2)$ is called a transition map. Note that transition maps are homeomorphisms of subsets of $\mathbb{C}$.

Definition 3.1.1. A Riemann surface is an atlas where all the transition maps are holomorphic$^1$.

Example 3.1.2.  
0) $\mathbb{C}$ with an atlas containing only the identity $id : \mathbb{C} \to \mathbb{C}$.

1) $\Omega \subset \mathbb{C}$ with an atlas containing only the inclusion.

2) $\hat{\mathbb{C}} =$ the one point compactification of $\mathbb{C}$. The atlas consists of two charts $id : U = \mathbb{C} \to \mathbb{C}$ and $id : V = \mathbb{C} \to \mathbb{C}$ with the transition map $\phi : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ defined by $\phi(z) = \frac{1}{z}$.

3) Let $f : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic. One can endow $f^{-1}(0)$ with a complex structure$^2$.

Notation 3.1.3. We will always denote by $S$ a topological surface and by $X$ a surface with a complex structure (a Riemann surface).

---

$^1$Note that since they are also homeomorphisms they are actually biholomorphic.

$^2$"inverse function thm"?
Definition 3.1.4. Let $X$ be a Riemann surface. A map $f : X \to \mathbb{C}$ is holomorphic if for every chart $(U, \phi)$ the map $f \circ \phi^{-1}$ is holomorphic.

The morphism $f : X \to Y$ is holomorphic if for every chart $(U, \phi)$ of $X$ and every chart $(V, \psi)$ of $Y$ the map $\psi \circ f \circ \phi^{-1}$ is holomorphic (where defined).

In the special case that $f : X \to \hat{\mathbb{C}}$ is holomorphic, we say that $f$ is meromorphic.

Fact 3.1.5. From the course on complex analysis we know that $\text{Mero}(\hat{\mathbb{C}})$ is the set of rational functions.

Definition 3.1.6. Given a Riemann surface $X$, $\text{Aut}(X)$ is the set of holomorphic maps $f : X \to X$ which are homeomorphisms. This is in fact a group since a composition of holomorphic maps is holomorphic, and as for the inverse: if $f$ is holomorphic and injective then $f'(z) \neq 0$, we can find a holomorphic inverse for $f$ in a neighborhood of $z$. Collecting the charts we define $f^{-1}$.

Definition 3.1.7. $\text{SL}_2(\mathbb{C})$ is the set of $2 \times 2$ matrices with complex entries and determinant equal to 1. $\text{PSL}_2(\mathbb{C}) = \text{SL}_2(\mathbb{C}) / \sim$ where $A \sim B$ if they differ by a non-zero (complex) scalar.

Theorem 3.1.8. $\text{Aut}(\hat{\mathbb{C}}) = \text{PSL}_2(\mathbb{C})$ identified via the isomorphism

$$
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} \mapsto \frac{az + b}{cz + d}
$$

Theorem 3.1.9 (The Uniformization Theorem). If $X$ is a simply connected Riemann Surface, then $X$ is biholomorphic to $\mathbb{C}, \hat{\mathbb{C}}$ or the unit disc $\Delta = \{z| |z| < 1\}$.

Remark. Notice that $\hat{\mathbb{C}}$ is not homeomorphic to $\mathbb{C}$ or $\Delta$, therefore it is different as a Riemann surface. Liouville’s theorem implies that any holomorphic map from $\mathbb{C}$ to $\Delta$ must be constant, thus it cannot be a homeomorphism. Therefore $\mathbb{C}, \hat{\mathbb{C}}, \Delta$ are distinct Riemann surfaces.

We can compute $\text{Aut}(\Delta)$ and $\text{Aut}(\mathbb{C})$ relatively painlessly from Theorem 3.1.8, and

Theorem 3.1.10 (The Riemann Mapping Theorem). If $D \subseteq \hat{\mathbb{C}}$ is simply connected, and its complement contains more, than one point then there is a biholomorphic homeomorphism $h : D \to \Delta$.

If $\phi \in \text{Aut}(\mathbb{C})$, then it is a holomorphic function on $\hat{\mathbb{C}}$, with an isolated singularity only (possibly) at $\infty$. If the singularity was essential, or a pole of order greater than 1 then in a neighborhood of $\infty$, $\phi$ would not be injective. Thus $\phi$ must be a meromorphic function on $\hat{\mathbb{C}}$ which is in fact linear. Therefore,

$$
\text{Aut}(\mathbb{C}) = \{a + bz | a, b \in \mathbb{C}\}.
$$

As for $\Delta$, first recall by that by the Riemann Mapping Theorem $\Delta$ is biholomorphic to $\mathbb{H}^2 = \{z| \text{Im}(z) > 0\}$ the upper half plane. If $\phi \in \text{Aut}(\mathbb{H}^2)$, then by the Schwartz
Reflection Principle we can extend it to an automorphism of \( \hat{\mathbb{C}} \), which preserves the upper half plane. This is exactly the subgroup \( \text{PSL}_2(\mathbb{R}) \).

\[
\text{Aut}(\Delta) \cong \text{Aut}(\mathbb{H}^2) = \text{PSL}_2(\mathbb{R})
\]

Now we wish to classify all closed Riemann Surfaces \( X \). Any such surface will be orientable so \( X \) is homeomorphic to \( S^2 \), \( T^2 \), or \( S_g \) for \( g > 1 \). Since \( X \) is a Riemann surface - so is \( \tilde{X} \). By Theorem 3.1.9 there are three cases: \( \tilde{X} = \hat{\mathbb{C}}, \mathbb{C}, \Delta \).

If \( \tilde{X} = \hat{\mathbb{C}} \) then \( X \) is topologically a sphere so \( X = \hat{\mathbb{C}} \).

If \( \tilde{X} = \mathbb{C} \) then \( X \) is homeomorphic to a torus\(^3\) (if we allow open manifolds then \( X \) could also be an annulus). Here there are many different Riemann structures we can endow \( T^2 \) with. Given \( X \) and thinking of \( \pi_1(X) \) as automorphisms of \( \mathbb{C} \), i.e., \( z \rightarrow z + b_z + ib_y \), we have a homomorphism \( \pi_1(X) \hookrightarrow \mathbb{C} \). If we identify \( \pi_1(X) \) with \( \mathbb{Z}^2 \), this homomorphism is determined by \((1,0) \rightarrow b_1 \) and \((0,1) \rightarrow b_2 \) where \( b_1, b_2 \in \mathbb{C} \). So we get a parallelogram with vertices at \( 0, b_1, b_2, b_1 + b_2 \). It degenerates only if \( \frac{b_1}{b_2} \in \mathbb{R} \) (but that is impossible if \( X \) is a torus). Conversely, given \( b_1, b_2 \in \mathbb{C} \) with \( \frac{b_1}{b_2} \notin \mathbb{R} \) we get a torus with a complex structure \( X_{b_1,b_2} \) by identifying opposite sides of the parallelogram. When are \( X_{b_1,b_2} \) and \( X_{c_1,c_2} \) the same Riemann Surface? For example, if there exists a \( \lambda \in \mathbb{C} \) such that \( c_i = \lambda b_i \) then the automorphism

\[
\begin{align*}
  f & : \mathbb{C} \rightarrow \mathbb{C} \\
  z & \rightarrow \lambda z
\end{align*}
\]

descends to an automorphism \( f : X_{b_1,b_2} \rightarrow X_{c_1,c_2} \). Thus \( X_{b_1,b_2} \) is biholomorphic to \( X_{1,\lambda} \) where \( \lambda = \frac{b_1}{b_2} \). Notice that \( \lambda \notin \mathbb{R} \). Also note that \( X_{1,\lambda} \) is biholomorphic to \( X_{1,-\lambda} \) after switching orientation. In conclusion, after fixing the orientation one can parameterize \( X_{b_1,b_2} \) by \( \mathbb{H}^2, \) the upper half plane.

If \( X_0, X_1 \) are bi-holomorphic with parameters \( \lambda_1, \lambda_2 \) respectively, does it follow that \( \lambda_1 = \lambda_2 \)? No. The problem lies when we identified \( \pi_1(X) \) with \( \mathbb{Z}^2 \). If we had made a different choice we’d have gotten a different parameter. We fix this by making our choice of generators explicit.

Let \( T^2 \) be an oriented torus. Fix generators for \( \pi_1(T^2) \) so that their order matches with the orientation. Consider the set of pairs \( (X, f) \) so that \( X \) is a Riemann surface on the torus and \( f : T^2 \rightarrow X \) is an orientation preserving homeomorphism. There is a map from the set \( \{(X, f)\} \) to the upper half plane: Given \( (X, f) \), lift the paths \( \alpha, \beta \) that represent the generators in \( \pi_1(X) \) and post-compose with an automorphism which takes the endpoint of \( \tilde{\alpha} \) to 1. Then \( \lambda \) is the endpoint of the lift \( \tilde{\beta} \).

We’ve seen that the map \( (X, f) \rightarrow \lambda \) is onto, since given \( \lambda \) we can glue opposite sides of the parallelogram with vertices at \( 0, 1, \lambda, \lambda + 1 \) to get a Riemann surface on the torus \( X_{\lambda} \).

\(^3\)In fact if \( X \) is homeomorphic to the torus then \( \tilde{X} = \mathbb{C} \). The proof involves understanding the discrete abelian subgroups of \( \text{PSL}_2(\mathbb{R}) \).
We define \((X_0, f_0) \sim (X_1, f_1)\) if there exists a bi-holomorphic map \(\phi : X_0 \to X_1\) such that \(\phi \circ f_0\) is homotopic to \(f_1\).

## 3.2 Special Case: the Torus

We consider

\[
T^2 = S^1 \times S^1 = \mathbb{R}^2 / \langle (x, y) \mapsto (x + 1, y), (x, y) \mapsto (x, y + 1) \rangle.
\]

Let \((X, f)\) be a marked Riemann surface on \(T^2\). We have

\[
\mathbb{R}^2 \xrightarrow{\tilde{f}} \tilde{X} \xrightarrow{\tilde{\phi}} \mathbb{C} \xrightarrow{\phi} \mathbb{C} \xrightarrow{f} X
\]

where \(\tilde{X}\) is the universal cover of \(X\) and it is not canonically isomorphic to \(\mathbb{C}\). Indeed, we can choose \(\tilde{f}\) so that \(\tilde{f}(0, 0) = 0\) and \(\tilde{f}(1, 0) = 1\). Then \(\tilde{f}(0, 1) = \tau\) and if we put an orientation on \(X\), we can take \(\text{Im}(\tau) > 0\).

We have the following

**Lemma 3.2.1.** Let \((X_0, f_0)\) and \((X_1, f_1)\) be two marked Riemann surfaces with parameters \(\tau_0\) and \(\tau_1\) respectively. 

\((X_0, f_0) \sim (X_1, f_1)\) if and only if \(\tau_0 = \tau_1\).

**Proof.** First, we suppose \((X_0, f_0) \sim (X_1, f_1)\) and so there exists a conformal map \(\phi : X_0 \to X_1\) such that \(\phi \circ f_0 \sim f_1\). Since \(\phi \circ f_0 \sim f_1\), \((\phi \circ f_0)_* = (f_1)_*\) as maps \(\pi_1(T^2, x_0) \to \pi_1(X_1, f_1(x_0))\) (we may suppose that \(\phi(f_0(x_0)) = f_1(x_0)\)). We consider the following diagram

\[
\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{\tilde{f}} & \mathbb{C} \\
\downarrow & & \downarrow \\
T^2 & \xrightarrow{f} & X
\end{array}
\]

and without loss of generality, we suppose that \(\tilde{\phi}(0) = 0\), that is, \(\phi(z) = az\) for some complex number \(a\).

Since \(\tilde{f}_0(0, 0) = 0 = \tilde{f}_1(0, 0)\) and \(\tilde{f}_0(1, 0) = 1 = \tilde{f}_1(1, 0)\), we have \(\tilde{\phi}(1) = a = 1\) and so \(\tilde{\phi}(\tau_0) = \tau_1\).

On the other hand, \(id \circ \tilde{f}_0 \sim \tilde{f}_1\) in an equivariant way. Let \(\tilde{\phi}\) be the linear homotopy and define \(\phi\) to be the map induced by \(\tilde{\phi}\).

Hence, there is a bijection from equivalence classes of marked Riemann surfaces on \(T^2\) to the Upper Half Plane.

For any \(\tau = a + ib\), we can define \((X_\tau, f_\tau)\) by defining first \(\tilde{f}_\tau : \mathbb{R}^2 \to \mathbb{C}\),

\[
\tilde{f}_\tau(x, y) = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]
and then considering the map \( f_\tau \) induced by \( \tilde{f}_\tau \) so that the diagram

\[
\begin{array}{c}
\mathbb{R}^2 \xrightarrow{\tilde{f}_\tau} C \\
\downarrow \quad \downarrow \\
T^2 \xrightarrow{f_\tau} X_\tau
\end{array}
\]

commutes.

Now, let \( \begin{pmatrix} s & q \\ r & p \end{pmatrix} \in PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \) and define \( \tilde{f} : \mathbb{R}^2 \to C \) as

\[
\tilde{f}(x, y) = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \begin{pmatrix} s & q \\ r & p \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

Then, we consider the map \( f \) induced by \( \tilde{f} \) so that the diagram

\[
\begin{array}{c}
\mathbb{R}^2 \xrightarrow{\tilde{f}} C \\
\downarrow \quad \downarrow \\
T^2 \xrightarrow{f} X_\tau
\end{array}
\]

commutes. If \( z_0 := \tilde{f}(1, 0) \) and \( z_1 := \tilde{f}(0, 1) \), then

\[
\frac{z_1}{z_0} = \frac{pr + q}{r\tau + s},
\]

where \( \tau \) is the parameter associate to \( X_\tau \).

So we defined an action of \( SL_2(\mathbb{Z}) \) on \( \{(X, f)\} \).

Define \( SL_2(\mathbb{Z}) \) is the so called Mapping Class Group of \( T^2 \).

We denote by \( Diff^+(T^2) \) the group of orientation-preserving diffeomorphisms of \( T^2 \) and by \( Diff_0^+(T^2) \) the subgroup of diffeomorphisms of \( T^2 \) homotopic to the identity.

**Definition 3.2.2.** The Mapping Class Group of the Torus is denoted by \( MCG(T^2) \) and it is defined as

\[
MCG(T^2) := \frac{Diff^+(T^2)}{Diff_0^+(T^2)}.
\]

We have the following

**Lemma 3.2.3.** Let \( f_0, f_1 : T^2 \to T^2 \in Diff^+(T^2) \), then \( f_0 \sim f_1 \leftrightarrow (f_0)_* = (f_1)_* \).

*Proof.* If \( f_0 \sim f_1 \), then obviously \( (f_0)_* = (f_1)_* \).

If \( (f_0)_* = (f_1)_* \), first we lift the maps to the universal cover and we take the linear homotopy. Then we project down the homotopy and we get \( f_0 \sim f_1 \). \( \square \)

Hence, \([f_0] = [f_1]\) in \( MCG(T^2) \) if and only if (by definition) \( f_0 \sim f_1 \) if and only if (by the lemma) \( (f_0)_* = (f_1)_* \).

**Fact 3.2.4.** Every automorphism of \( \pi_1(T^2) = \mathbb{Z}^2 \) is the induced map of some diffeomorphism of \( T^2 \).
Indeed, we have $\text{Aut}(\mathbb{Z}^2) = GL_2(\mathbb{Z})$ and $\text{Aut}^+(\mathbb{Z}^2) = SL_2(\mathbb{Z})$ and for every $M \in GL_2(\mathbb{Z})$ we have the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{R}^2 & \overset{M}{\longrightarrow} & \mathbb{R}^2 \\
\downarrow & & \downarrow \\
T^2 & \longrightarrow & T^2.
\end{array}
\]

Hence, $\text{MCG}(T^2) = SL_2(\mathbb{Z})$.

$\text{MCG}(T^2)$ acts on marked Riemann surfaces of the torus in the following way: if $\varphi \in \text{MCG}(T^2)$ and $(X, f)$ is a marked Riemann surface of the torus, then $(X, f) \mapsto (X, f \circ \varphi^{-1})$. $(X, f \circ \varphi^{-1})$ is a marked Riemann surface since $(X, f) \sim (X_1, f_1) \Leftrightarrow (X, f \circ \varphi^{-1}) \sim (X_1, f_1 \circ \varphi^{-1})$.

We can sum up what we have proved in the case of the torus.

We have a bijection between $\mathbb{H}^2 := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$ and the Teichmüller space $\mathcal{T}(T^2)$ for $T^2$.

The Mapping Class Group $\text{MCG}(T^2)$ acts on $\mathcal{T}(T^2)$ in the following way: if $\tau \in \mathcal{T}(T^2)$ and $\varphi = \left( \begin{array}{cc} p & q \\ r & s \end{array} \right) \in \text{MCG}(T^2) = SL_2(\mathbb{Z})$ so $\varphi^{-1} = \left( \begin{array}{cc} s & -q \\ -r & p \end{array} \right)$ and

\[
\varphi \cdot \tau = \frac{p\tau - q}{-r\tau + s}.
\]

Let $(X_0, f_0)$ and $(X_1, f_1)$ marked Riemann surfaces and $\varphi : X_0 \rightarrow X_1$. If $(\varphi \circ f_0) \sim f_1 \Leftrightarrow (X_0, f_0) \sim (X_1, f_1)$, then there exists $\psi \in \text{MCG}(T^2)$ such that $\varphi \circ f_0 \sim f_1 \circ \psi^{-1}$.

**Definition 3.2.5.** The quotient of $\mathcal{T}(T^2)$ under the action of $\text{MCG}(T^2)$ is called the Moduli Space $\mathcal{M}_1$ of the torus and is the set of unmarked conformal structure of $T^2$.

Since

\[
\mathcal{M}_1 = \frac{\mathcal{T}(T^2)}{\text{MCG}(T^2)} = \frac{\mathbb{H}^2}{\text{PSL}_2(\mathbb{Z})},
\]

the Moduli Space of the torus is in bijection with the region in Figure 3.1.

### 3.3 Quasi-Conformal Map on the Torus

Let $f : X_0 \rightarrow X_1$ a diffeomorphism between manifolds.

**Definition 3.3.1.** $f$ is a $K$-quasi-conformal map if for all charts $(U, \phi)$ on $X_0$ and $(V, \psi)$ on $X_1$, the composition $\psi \circ \phi \circ \psi^{-1}$ is $K$-quasi-conformal where it is defined.

We want to define the Teichmüller distance in $\mathcal{T}(T^2)$.

**Definition 3.3.2.** Let $K_f$ be the quasi-conformal constant for $f$. The Teichmüller distance between the marked Riemann surfaces $(X_0, f_0)$ and $(X_1, f_1)$ is

\[
d_{\text{Teich}}((X_0, f_0), (X_1, f_1)) := \frac{1}{2} \log(\text{inf}(K_f)),
\]

where the infimum is taken over the diffeomorphisms $\phi : X_0 \rightarrow X_1$ with $\phi \circ f_0 \sim f_1$. 

This distance determines a metric on $\Xi(T^2)$ which is equal to the hyperbolic metric on $\mathbb{H}^2$ and this is the reason of why we have $\frac{1}{2}$ in the definition of the Teichmüller distance.

The infimum is realized in the case of the torus, but it is not realized in general. Moreover, we will prove that when the infimum is realized, the diffeomorphism that realizes the infimum is unique and "behaves nicely", in the sense that it is a stretch in one direction and a shrink in the other direction.

In the case of the torus we have this nice result which is not true in the general case.

**Theorem 3.3.3.** Let $X$ be a conformal structure in $T^2$. Given any two points $z_0$ and $z_1$ in $X$, there exists a conformal map $\phi : X \to X$ isotopic to the identity such that $\phi(z_0) = z_1$.

**Proof.** We lift $X$ to the universal cover $\tilde{X} \cong \mathbb{C}$ and we define

$$\tilde{\phi}_t(z) = z + (\tilde{z}_1 - \tilde{z}_0)t,$$

where $\tilde{z}_0$ and $\tilde{z}_1$ are two lifts of $z_0$ and $z_1$. Then we project $\tilde{\phi}_1$ down. By construction, this map satisfies the thesis of the theorem. 

**3.4 Hyperbolic and Conformal Structures**

We will define the hyperbolic and conformal structures on a manifold and we will prove that conformal structures, Riemann surface structures, and hyperbolic structures all coincide when the dimension of the manifold equals 2 and $genus \geq 2$. 
3.4.1 Hyperbolic Structures

**Definition 3.4.1.** (First Definition) A hyperbolic structure on a manifold $M$ is a Riemannian metric with constant sectional curvature equal to $-1$.

**Theorem 3.4.2.** There is a unique simply connected $n$-manifold $(\mathbb{H}^n)$ with a complete hyperbolic structure.

An important fact is that the isometry group of $\mathbb{H}^n$ acts transitively. In other words, if $x, y \in \mathbb{H}^n$, then there exists an isometry $\phi : \mathbb{H}^n \to \mathbb{H}^n$ with $\phi(x) = y$. This shows that $\mathbb{H}^n$ is a homogeneous space (every point looks like every other point).

**Corollary 3.4.3.** Let $M$ be a complete hyperbolic $n$-manifold. Then every point $x \in M$ has a neighborhood isometric to a neighborhood of $\mathbb{H}^n$.

**Definition 3.4.4.** (Second Definition) A hyperbolic structure on a manifold $M$ is an atlas with chart maps mapping into $\mathbb{H}^n$ and transition maps given by restrictions of isometries of $\mathbb{H}^n$.

At this point the focus will shift to the case where $n = 2$. In this case $\mathbb{H}^2 \subset \mathbb{C}$ and isometries are holomorphic. This immediately implies that a hyperbolic structure determines a Riemann surface structure.

**Lemma 3.4.5.** Let $X^h \to X^r$ be the natural map from the space of hyperbolic structures to the space of Riemann surface structures. Then it is the case that $X^r_0 = X^r_1$ if and only if $X^h_0 = X^h_1$.

**Proof.** ($\Leftarrow$) This direction is obvious.

($\Rightarrow$) Since $X^r_0$ and $X^r_1$ also carry a hyperbolic structure, we see by Theorem 3.4.2 that the universal cover of $X^r_0$ and $X^r_1$ is $\mathbb{H}^2$ (specifically we will use the disc model $\Delta$).

\[
\begin{array}{c}
\Delta = \tilde{X}^r_0 \xrightarrow{\tilde{\phi}} \tilde{X}^r_1 = \Delta \\
\downarrow \quad \downarrow \\
X^r_0 \quad X^r_1
\end{array}
\]

Now there is a map conformal automorphism $\tilde{\phi} : \Delta \to \Delta$. We saw earlier that conformal automorphisms of $\Delta$ are hyperbolic isometries. Thus the map $\tilde{\phi}$ descends to a hyperbolic isometry $\phi : X^r_0 \to X^r_1$. 

$\square$
At this point we would like to see which Riemann surfaces carry a hyperbolic structure explicitly. To do this, we first recall the Uniformization Theorem which states that every simply connected Riemann surface is either $\mathbb{C}$, $\hat{\mathbb{C}}$, or $\Delta = \mathbb{H}^2$. Knowing the classification of surfaces and the fact that all Riemann surfaces are orientable (holomorphic maps are orientation-preserving) we can classify closed Riemann surfaces by their universal covers.

**Theorem 3.4.6.** Let $X$ be a closed Riemann surface.

1. $\tilde{X} = \hat{\mathbb{C}}$ if and only if $X = \hat{\mathbb{C}}$.
2. $\tilde{X} = \mathbb{C}$ if and only if $X \cong T^2$.
3. $\tilde{X} = \Delta$ if and only if $X$ has genus $\geq 2$.

**Proof.** The only orientable surface covered by the sphere is the sphere. When combined with the fact that every conformal automorphism of $\hat{\mathbb{C}}$ has a fixed point (implies group of deck transformations is trivial) gives the first statement.

The deck transformation group ($z \mapsto az + b$ such that there are no fixed points) of $\tilde{X} = \mathbb{C}$ is abelian. The only surfaces with abelian fundamental group are $T^2$ and $\hat{\mathbb{C}}$, but $\hat{\mathbb{C}}$ is not covered by the plane which proves the forward implication of 2).

The backward implication of 3) holds since these surfaces have non-abelian fundamental group. This only leaves the backward implication of 2) and the forward implication of 3) which are equivalent.

To do this we need to examine abelian subgroups of $PSL_2(\mathbb{R}) = Isom^+(\mathbb{H}^2)$.

**Exercise 3.4.7.** Let $\gamma, \beta \in Isom^+(\mathbb{H}^2)$. $[\gamma, \beta] = id$ if and only if $fix(\gamma) = fix(\beta)$ where $fix(\gamma) = \{z \in \Delta | \gamma(z) = z\}$.

By the classification of isometries we get three cases.

1. $fix(\gamma) = z \in \Delta$ if and only if $\gamma$ is elliptic. But deck transformations have no fixed points.
2. $fix(\gamma) = z \in \partial\Delta$ if and only if $\gamma$ is parabolic. Another exercise is if $g_0$ and $g_1$ are parabolic then there exists an $h$ such that $h \circ g_0 \circ h^{-1} = g_1$.

In the upper half space model $\mathbb{H}^2$ we can assume by the exercise that $\gamma(z) = z + 1$. Thus if $[\gamma, \beta] = id$ then $\beta(z) = z + \lambda$ with $\lambda \in \mathbb{R}$.

**Exercise 3.4.8.** Any injective map of $\mathbb{Z}^2$ in $\mathbb{R}$ has dense image.

If $X$ is a Riemann surface and $\tilde{X} = H$ then $\pi_1(X) \hookrightarrow PSL_2(\mathbb{R})$. The image must also be discrete or else it is not a deck transformation of $\mathbb{H}^2$. In fact the same problem comes about with elliptics.

3. $fix(\gamma) = fix(\beta) = \{z_0, z_1\} \subset \partial\Delta$. There is a unique geodesic $\Lambda \subset \Delta$ with endpoints $z_0$ and $z_1$. The set of orientation-preserving isometries preserving $\Lambda$ looks like $\mathbb{R}$ with the additive structure.
These arguments show that there are no discrete subgroups of $PSL_2(\mathbb{R})$ isomorphic to $\mathbb{Z}^2$. Thus the torus $T^2$ cannot be covered by $\Delta$ which finishes the 2 missing implications.

**Lemma 3.4.9.** If $X^r$ is a Riemann surface and $\tilde{X} = \Delta$ then $X^r$ has a hyperbolic structure $X^h$ with $X^h = X^r$.

**Proof.** Since $\tilde{X} = \Delta$, around any point $p \in X^r$ we can find a neighborhood that lifts to the universal cover. This neighborhood and the lift together form a chart around the point $p$ mapping into $\Delta$. The only thing left to check is that the transition maps are hyperbolic isometries. Given two charts of the above form $p \in (U, \varphi)$ and $q \in (V, \psi)$ it might be the case that we choose lifts that do not intersect. But there is a deck transformation taking one lift of $U$ to the lift that intersects the chosen lift of $V$. This transformation is, by definition, a hyperbolic isometry.

This discussion gives an extension of the Uniformization Theorem.

**Theorem 3.4.10.** If $X$ is a closed Riemann surface with $\text{genus}(X) \geq 2$, then $X$ has a hyperbolic structure. If $\text{genus}(X) = 1$, then $X$ has a Euclidean structure. If $\pi_1(X) = \mathbb{Z}, \{1\}$ is abelian, then $\tilde{X} = \Delta$ or $\tilde{X} = \mathbb{C}$. If $\pi_1(X)$ is non-abelian, then $\tilde{X} = \Delta$.

### 3.4.2 Conformal Structures

We would like to define a notion of angle.

In $\mathbb{R}^n$ with the usual dot product structure the regular notion of angle $\theta = \angle(v, w)$ is given by $\langle v, w \rangle = |v||w|\cos\theta$. If $V$ is a vector space with an inner product $\langle , \rangle$, then define $\theta = \angle(v, w)$ to be

$$\langle v, w \rangle = |v||w|\cos\theta.$$

Two inner products $\langle , \rangle_0$ and $\langle , \rangle_1$ are **conformally equivalent** if $\angle_0(v, w) = \angle_1(v, w)$ for all $v, w \in V$.

**Exercise 3.4.11.** $\langle , \rangle_0$ and $\langle , \rangle_1$ are conformally equivalent if and only if there exists $\lambda > 0$ such that $\langle , \rangle_0 = \lambda \langle , \rangle_1$.

**Definition 3.4.12.** A **conformal structure** on $V$ is an equivalence class of inner products.

**Definition 3.4.13.** Let $g_0, g_1$ be Riemannian metrics on $M$. Then $g_0, g_1$ are conformally equivalent if there exists $\lambda : M \rightarrow \mathbb{R}^+$ such that $g_0 = \lambda g_1$ and $\lambda$ is smooth.

**Example 3.4.14.** Let $H = \{z \in \mathbb{C}|\text{Im}(z) > 0\}$ and $\lambda : H \rightarrow \mathbb{R}$ is given by $\lambda(z) = \frac{1}{\text{Im}(z)}$. If $g_\mathbb{E}$ is the Euclidean metric on $H \subset \mathbb{C}$ then the hyperbolic metric on $H$ denoted $g_{\mathbb{H}}$ is conformally equivalent to $g_\mathbb{E}$ since $g_{\mathbb{H}} = \lambda g_\mathbb{E}$. 
3.5. **$T(S)$ IS HOMEOMORPHIC TO $\mathbb{R}^{6g-6}$**

**Definition 3.4.15.** A conformal structure on a manifold $X$ is an equivalence class of Riemannian metric.

**Theorem 3.4.16.** (Isothermal Coordinates due to Gauss) Let $g$ be a Riemannian metric on $\Omega \subset \mathbb{C} = \mathbb{R}^2$ then there exists a diffeomorphism $\phi : \Omega \to \Omega'$ such that $\phi^*g = \lambda g_E$ where $\lambda : \Omega \to \mathbb{R}^+$. 

Now assume that $X$ has a conformal structure and let $g$ be a metric in the conformal class. Let $(U, \phi)$ be a chart and let $\phi^*g$ be the metric on $\phi(U)$. Using isothermal coordinates, we can assume that 

$$\phi^*g = \lambda g_E$$

defines an atlas. Check that transition maps are holomorphic which will give a Riemann surface structure on $X$, and check that the Riemann surface structure does not depend on the choice of $g$. This shows that conformal structures, Riemann surfaces, and hyperbolic structures all coincide when the dimension of the manifold equals 2 and genus $\geq 2$.

### 3.5 $T(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$

Today, we think of $T(S)$ as the space of hyperbolic structures on $S$. We aim to prove:

**Theorem 3.5.1.** $T(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$ where $g$ is the genus of the surface $S$.

### 3.6 Length functions

**Definition 3.6.1.** A map $\gamma : \mathbb{S}^1 \to X$ is an essential curve if $\gamma$ is not homotopic to the constant map.

**Lemma 3.6.2.** Let $\gamma$ be an essential curve in $X$, either:

1. $\gamma$ is homotopic to $\gamma'$ where $\text{Im}\gamma'$ is a geodesic in $X$, or

2. $\forall \varepsilon > 0$ there is a $\gamma_\varepsilon$ homotopic to $\gamma$ such that the length of $\gamma_\varepsilon$ in $X$ is smaller than $\varepsilon$.

**Proof.** We will see that if $X$ is closed (compact with no boundary) case 2 is impossible. Now $\gamma : \mathbb{S}^1 \to X$ is not homotopically trivial, therefore $\gamma_\ast(\pi_1(\mathbb{S}^1))$ is isomorphic to $\mathbb{Z}$. Let $X_\gamma$ be the corresponding covering space. $X_\gamma$ is topologically an annulus. Metrically, there is an isometry $\phi$ of $\mathbb{H}^2$ such that $X_\gamma = \mathbb{H}^2/ < \phi >$. $\phi$ is either hyperbolic or parabolic:

1. If $\phi$ is a hyperbolic isometry, let $\tilde{\gamma}'$ be the (unique) axis for $\phi$. Then $\tilde{\gamma}'$ descends to a closed loop $\gamma'$ in $X_\gamma$ which is a local geodesic. In $X_\gamma$: $\gamma_\ast = \gamma'_\ast$ so there is a homotopy from $\gamma$ to $\gamma'$ which descends to $X$. 

2. If $\phi$ is a parabolic isometry, then $\text{Im}\gamma'$ is a geodesic in $X_\gamma$. In $X_\gamma$: $\gamma_\ast = 0$ so there is a homotopy from $\gamma$ to a constant map which descends to $X$. 


2. If \( \phi \) is a parabolic isometry \( X_\gamma \) is as in Figure 3.2 and \( \gamma \) is the core curve which is homotopic to curves of arbitrarily small loops.

Now we are in a position to define length functions:

**Definition 3.6.3.** Let \( \gamma \) be an essential closed loop on \( S \), the *length function* corresponding to \( \gamma \) is a map \( l_\gamma : \mathcal{T}(S) \to [0, \infty) \) defined by \( l_\gamma(X, f) \) is the length of the geodesic homotopic to \( f \circ \gamma \) in the hyperbolic structure \( X \) or 0 if no such representative exists.

**Exercise 3.6.4.** Show that this is a well defined function on \( \mathcal{T}(S) \), i.e., if \( (X, f) \sim (Y, g) \) in \( \mathcal{T}(S) \), then \( l_\gamma(X, f) = l_\gamma(Y, g) \).

**Lemma 3.6.5.** If \( S \) is closed, then \( l_\gamma(X, f) > 0 \) for every \( (X, f) \in \mathcal{T}(S) \), and every essential curve \( \gamma \).

**Proof.** We define the injectivity radius of a hyperbolic surface at a point: v

**Definition 3.6.6.** Let \( X \) be a hyperbolic structure on \( S \), and \( p \in X \). If \( D = \text{diam}(X) \) then \( B(p, D) \) is \( X \), on the other hand, since the exponential function is injective in a small enough neighborhood of \( p \), there is some \( \varepsilon > 0 \) such that \( B(p, \varepsilon) \) is homeomorphic to a disc. The injectivity radius of \( X \) at \( p \), denoted \( \text{inj}(X, p) \) is \( \sup \{ r \mid B(p, r) \text{ is homeomorphic to a disc} \} \).

We have defined a continuous map \( \text{inj} : X \to (0, \infty) \). Since \( X \) is compact, it takes on a maximum and a minimum. The minimum of this function is denoted \( \text{inj}(X) \). Notice that for any closed loop in \( X \), \( l_\gamma(X) \geq 2\text{inj}(X) \) and we get the lemma.

**Definition 3.6.7.** A *pair of pants* is a surface homeomorphic to \( \mathbb{S}^2 \setminus 3 \) discs.

**Facts about a Pair of Pants \( \mathcal{P} \):**
3.6. LENGTH FUNCTIONS

Figure 3.3: A pair of pants is homeomorphic to a thrice punctured sphere.

1. \( \chi(\mathcal{P}) = -1 \)

2. Every essential simple closed curve (ssc in the sequel) is isotopic to a component of the boundary (One way to see this is to think of the curve as a Jordan curve in \( S^2 \setminus \{x_1, x_2, x_3\} \) the curve separates this space into two components and partitions \( \{x_1, x_2, x_3\} \). The partition tells us which boundary component the curve is homotopic to).

3. If \( \phi : \mathcal{P} \to \mathcal{P} \) is an orientation preserving homeomorphism which doesn’t permute the boundary components then \( \phi \) is homotopic to the identity. See Figure 3.4.

Figure 3.4: The fundamental group of \( \mathcal{P} \) is \( F_2 = \langle \alpha, \beta \rangle \) where \( \alpha \) is the red curve and \( \beta \) is the blue curve. Since \( \phi \) doesn’t permute boundary components, \( \alpha, \phi(\alpha) \) are freely homotopic to the same boundary curve so they are homotopic. Similarly, \( \beta, \phi(\beta) \) are homotopic. Thus \( \phi \) is homotopically trivial.

Fact: If \( \phi : S \to S \) is homotopic to the identity, then it is isotopic to the identity. If \( \gamma_1, \gamma_2 \) are s.c.c which are homotopic, then they are isotopic. (see [6]).

Proposition 3.6.8. Given \( a, b, c \geq 0 \), there exists a unique \( X \) hyperbolic structure on \( \mathcal{P} \) where the boundary components are geodesics with hyperbolic lengths \( a, b, c \) respectively.
Proof A of proposition 3.6.8. We will prove:

**Lemma 3.6.9.** Given \( a', b', c' > 0 \) there is a unique hyperbolic right angled hexagon \( H \) with alternating side lengths \( a', b', c' \).

Given \( a, b, c \), take two copies of \( H(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}) \): \( H_1, H_2 \). Glue the sides with the unspecified edge lengths, to get a space which is topologically a pair of pants \( X \). \( X \) is in fact the two sheeted (metric) covering of \( H_1 \) and thus is itself a hyperbolic surface. Since lifts of boundary components are geodesics in the upper half plane - the boundary is geodesic, and the components have the specified lengths.

Uniqueness: Let \( X \) be a hyperbolic pair of pants with boundary components \( A, B, C \) with lengths \( a, b, c \). Let \( \delta_{AB} \) be a geodesic arc connecting \( A, B \) which is perpendicular to both components (to find such an arc - lift \( A, B \) to \( \mathbb{H}^2 \) and consider the shortest arc between them. By a surgery argument one can see that it will be perpendicular to both geodesics). Let \( \delta_{BC} \) and \( \delta_{AC} \) be geodesic arc perpendicular to \( B, C \) and \( A, C \). Cutting along \( \delta_{AB}, \delta_{BC} \) and \( \delta_{AC} \) we get two right angled hexagons. Moreover, they will have three alternating side lengths which are equal - namely the ones that we don’t specify. Thus, by the uniqueness of \( H(a', b', c') \) those hexagons are isometric. So \( X \) can be constructed by gluing two isometric right-angled hexagons of appropriate alternating side lengths. Uniqueness now follows from the uniqueness of the hexagons. \( \square \)

**Proof of Lemma 3.6.9.** Figure 3.5 shows how to construct a pentagon with specified alternating side lengths \( a, b \).

---

**Figure 3.5:** The green geodesics are the ones whose length is specified.

Start with the geodesic line \( L_1 \) and draw the (non-geodesic) line \( M \) such that every point on \( M \) is a hyperbolic distance \( b \) away from \( L_1 \). Pick any point \( x \) and
draw a geodesic segment of length $a$ starting at $x$ and perpendicular to $L_1$. Let $L_2$ be a common perpendicular to this segment and the real line which intersects the real line at point $y$. Let $L_3$ be a geodesic with endpoint at $y$ and tangent to $M$ at point $z$. Complete the pentagon by drawing the shortest geodesic from $z$ to $L_1$ (its length is $b$). We get a geodesic pentagon with alternating side lengths $a, b$.

We alter this construction slightly to get a hexagon: move $L_3$ away from $L_2$ so that the distance on the real line between them is $r$. I.e., $L_3(r)$ is a Euclidean circle perpendicular to the boundary at distance $r$ from $L_2$ which is tangent to $M$ (see Figure 3.6). Denote the tangency point $z$. The last side of the hexagon still has length $b$. Let $t(r)$ be the hyperbolic distance between $L_2$ and $L_3(r)$. $t(0) = 0$ and $\lim_{r \to \infty} t(r) = \infty$. It is also not hard to see that $t$ is a strictly increasing function. Therefore, by the intermediate value theorem, there is an $r$ such that $t(r) = c$. Thus we get a hexagon with side lengths $a, b, c$.

![Figure 3.6: The green geodesics are the ones whose length is specified.](image)

Uniqueness of the hexagon. Suppose we have another such hexagon. By composing with a Mobius transformation we can assume that the left endpoint of the edge of length $a$ is at $x$ and the edge to the right of it is a segment of $L_1$ - the $y$ axis. Uniqueness now follows from the fact that $t(r)$ is strictly monotone increasing.

We have constructed a polygon as needed in the case where $a, b, c > 0$ and where $a, b > 0$ and $c = 0$. The case where $a > 0$ and $b = c = 0$ is left to the reader.

Proof B of proposition 3.6.8. Consider a hyperbolic isometry $z \to e^\lambda z$ which takes the vertical line $x = 1$ to the line $x = e^\lambda$. The quotient is topologically an annulus. Figures 3.7 and 3.8 show how we can get different metric structures. Notice that in Figure 3.8 we get an incomplete metric structure. Indeed, the length of the green
Figure 3.7: When we glue without sheering we get a complete metric space.

Figure 3.8: When we glue with sheering we get an incomplete metric space. The length of the boundary curve depends on the amount of sheering. The bigger $r$ is - the smaller the boundary curve becomes.
The length of the boundary curve changes with $\lambda$. The length of the boundary curve is $\lambda < \infty$. The length of the boundary curve changes with $\lambda$.

$$\text{len} = l_2 = l_1/e^\lambda$$

$$\text{len} = l_1 = l_0/e^\lambda$$

$$\text{len} = l_0 = e^\lambda - 1$$

Figure 3.9: The length of the green curve is $< \infty$. The length of the boundary curve is $\lambda$.

Now consider two ideal triangles. An ideal triangle is one with all three vertices on the boundary. All ideal triangles are isometric. For each triangle, the midpoint of a side is the intersection point of the side with the circle inscribed inside the triangle. We glue the triangles together along their boundaries with an orientation reversing isometry which takes the midpoints $d, e, f$ to the points at a distance $x, y, z$ from $d, e, f$ respectively. We get a pair of pants. By changing $x, y, z$ we can control the side lengths of the boundary curves. In fact, the lengths of the boundary curves will be $|x + y|, |y + z|, |x + z|$. See Figure 3.11.

Given $a, b, c$ we can solve $|x + y| = a, |y + z| = b, |z + x| = c$ and perform the construction described above. Actually we get eight different solutions for $x, y, z$. Does that contradict the uniqueness of the hyperbolic structure? The answer is no by Lemma 3.6.10.

**Lemma 3.6.10.** Every hyperbolic pair of pants with positive boundary lengths has eight decompositions into two ideal triangles.

**Proof.** Pick two points from each boundary circle and connect them with non-intersecting paths. For each boundary circle, pick a direction and start twisting the circle in that direction. In the universal cover this amounts to sliding the points (equivariantly along the geodesic). In the limit (in the universal cover) we get an ideal triangle which descends to an ideal triangle downstairs (see Figure 3.12). The complement will also be an ideal triangle$^4$. So any hyperbolic structure on a pair of pants with positive boundary lengths can be gotten by gluing two ideal triangles. Notice that we had 8

$^4$why? Notice that the complement of the region we spun is also a hexagon.
Figure 3.10: We glue the triangles together along their boundaries with an orientation reversing isometry which takes the midpoints $d, e, f$ to the points at a distance $x, y, z$ from $d', e', f'$ respectively.

Figure 3.11: The length of the boundary curve is $|x + y|$. 
choices when twisting the boundary components which is exactly the descrepency in the existence proof. Thus given boundary lengths - the hyperbolic structure on the pair of pants is unique.

\[ \square \]

Figure 3.12: We spin the endpoints of the hexagon until we get a decomposition into two ideal triangles.

### 3.7 Fenchel-Nielsen Coordinates

Let \( S = S_g \) be a closed surface of genus \( g \geq 2 \). We recall that

\[
\mathcal{T}(S) := \{(X, f) | X \text{ is an hyperbolic surface } f : S \to X, f \in \text{Homeo}^+ \}/\sim,
\]

where \( (X_0, f_0) \sim (X_1, f_1) \) if there exists an isometry \( \phi : X_0 \to X_1 \) such that \( \phi \circ f_0 \sim f_1 \).

Here there is an outline of what we are going to prove in this section. As before, we consider cutting along disjoint simple closed curves (actually we consider geodesics with respect to the hyperbolic metric on \( X \)). When there are no more simple closed curves in the remaining open sets, then we will prove that we are left with pairs of pants. We proved that the hyperbolic structure of a pair of pant is completely determined by the lengths of the curves in the boundary of the pair of pant.

Since \( S \) is reconstructed by gluing all the pairs of pants together suitably, we can consider a system of coordiantes for the Teichmüller space \( \mathcal{T}(S) \) given by the set of lengths used in the above decomposition of \( S \) in pairs of pants and the set of the twist coefficients used to glue the pieces of the decomposition together.

This system of coordinates gives the homeomorphism

\[
\mathcal{T}(S) \cong \mathbb{R}^{6g-6}.
\]

We have the following definitions:

**Definition 3.7.1.** If \( X \) is a surface with boundary, we say that \( \gamma \) is peripheral if it is homotopic to a boundary component of \( X \).
Definition 3.7.2. The geometric intersection number of two homotopy classes of curves $\alpha$ and $\gamma$ is the minimum number of intersections between curves representing the two homotopy classes.

Let $\wp$ be a maximal collection of disjoint simple closed curves such that

1. If $\alpha, \beta \in \wp$, $\alpha \sim \beta$;

2. If $\gamma$ is an essential simple closed curve and $\gamma \notin \wp$, then there exists $\alpha \in \wp$ such that $i(\alpha, \gamma) > 0$.

Lemma 3.7.3. If $\chi(S) < 0$, then $S \setminus \wp$ is a collection of pants.

Proof. Let $X$ be a component of $S \setminus \wp$.
Since (2) holds, $\wp$ doesn’t contain essential curves and so $X \neq D^2$.
Moreover, $X \neq S^1 \times [0, 1]$ because otherwise there are two elements $\alpha, \beta \in \wp$, $\alpha \sim \beta$ and this contradicts (1).
By the classification of surfaces, if $X$ is not a pair of pants, then $X$ has an essential, non-peripheral simple closed curve.
By what we have just said, $X$ is a pair of pants. Since this is true for every component of $S \setminus \wp$, $S \setminus \wp$ is a collection of pants.

Hence, we get a so-called pants decomposition $\wp$. Examples of pants decomposition are shown in Figure 3.13 and Figure 3.14.

Recall that $\chi(P) = -1$, where $P$ is a pair of pants.
If $S \setminus \wp = P_1 \sqcup \ldots \sqcup P_k$, then $\chi(S) = -k = 2(1 - g)$ and so $k = 2(g - 1)$ Since the
number of pairs of pants is $2(g - 1)$, we have $3g - 3$ curves in $\varphi$.
Let $l_\varphi : \Xi(S) \to (\mathbb{R}^+)^{3g - 3}$ be the length function which sends an element $(X, f) \in \Xi(S)$ to the length of the curves in $\varphi$.
If $(X_0, f_0), (X_1, f_1) \in \Xi(S)$ have $l_\varphi(X_0, f_0) = l_\varphi(X_1, f_1)$, then there exists $\phi : X_0 \to X_1$ such that $\phi \circ f_0 \sim f_1$. The map $\phi$ takes curves in $\varphi$ to curves in $\varphi$ and the length of curves in $\varphi$ is equal on $X_0$ and $X_1$.
We have the following fact:

**Fact 3.7.4.** Let $P$ be a pair of pants. If $\phi : P \to P$ is a orientation-preserving homeomorphism that preserves the boundary components, then $\phi$ is homotopic to the identity.

Hence, we have an isometry between a pair of pants in $X_0$ and the correspondent pair of pants in $X_1$. The problem is that maybe we cannot extend this isometry to agree on the boundary of the pair of pants because there can be *twists* in the gluing of the boundary. We want to study what happens in more details.

**Theorem 3.7.5.** Let $\gamma_0$ and $\gamma_1$ be a collection of disjoint essential simple closed curves on $S$ such that the curves in $\gamma_0$ are homotopic to the curves in $\gamma_1$. Then there exists a homeomorphism $f : S \to S$ such that $f \sim id$ and $f(\gamma_0) = \gamma_1$.

This theorem is true even if the collection $\gamma_0$ contains homotopic curves. We refer to [6] for the proof of this Theorem.
Let $P$ be a pair of pants in $S \setminus \varphi$. Then there exists a hyperbolic structure $Y$ on $P$ and isometries $i_0 : Y \to X_0$ and $i_1 : Y \to X_1$ that map $Y$ to $P$.
By the Theorem 3.7.5, we can assume that $\phi$ takes geodesic representatives of $\varphi$ on $X_0$ to geodesic representatives of $\varphi$ on $X_1$. Hence, we have

$$
\begin{array}{c}
Y \\
\downarrow i_0 \\
X_0 \underbrace{\phi}_{i_1} \rightarrow X_1
\end{array}
$$

and the composition $i_1^{-1} \circ \phi \circ i_0$ is almost well-defined since if $x \in P$, then $\phi \circ i_0(x)$ is in the image of $i_1$. We want that $\phi \circ i_0(x)$ is the image of only one point of $i_1$ but we can have a problem to find this point if $x$ is on the boundary of $P$. It is easy to see that we can make $i_1^{-1} \circ \phi \circ i_0$ well-defined everywhere (see Figure 3.15).

We assume that $i_1^{-1} \circ \phi \circ i_0$ is well-defined. Since $i_1^{-1} \circ \phi \circ i_0 \sim id$, let $\psi_t$ be the homotopy such that $\psi_1 = id$, $\psi_0 = i_1^{-1} \circ \phi \circ i_0$. We define

$$
\phi_t(x) = \begin{cases} 
\phi(x), & \text{if } x \neq P \\
i_1^{-1} \circ \psi_1 \circ i_0(x), & \text{if } x \in P.
\end{cases}
$$

Notice that $\phi_t$ is not continuous.
What do we need to do to get continuity?

**Lemma 3.7.6.** Every orientation-preserving homeomorphism of $S^1$ is isotopic to the identity map.
Figure 3.15: The map $i_1^{-1} \circ \phi \circ i_0$ is well-defined since given a point $x$ in $P$, we can follow its image under this map and we know in which boundary component of $P$ it is going to land.

Since there exists $\varepsilon > 0$ such that the $\varepsilon$-neighborhood of each geodesic in $\varphi$ is an embedded annulus, we consider a collar $\varepsilon$-neighborhood for each curve in the boundary of the pairs of pants and we make $\varphi$ an isometry outside the collar neighborhood.

Proposition 3.7.7. Let $\varphi_0, \varphi_1 : X_0 \to X_1$ be isometries outside the $\varepsilon$-collars. Then there exists a homotopy $\phi_t$ such that $\phi_t = \varphi_0 = \varphi_1$ outside the collars.

Proof. The first part of the proof involves topology. We pick a base point $x_0$ in the boundary of one collar. We consider the universal covering $\pi : \mathbb{H}^2 \to \tilde{S}$ and we choose $\tilde{x}_0 \in \pi^{-1}(x_0)$. We lift $\tilde{\varphi}_0, \tilde{\varphi}_1 : \mathbb{H}^2 \to \tilde{S} = \mathbb{H}^2$ so that $\tilde{\varphi}_0(\tilde{x}_0) = \tilde{\varphi}_1(\tilde{x}_0)$.

Since $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$ are such that $\tilde{\varphi}_0(\tilde{x}_0) = \tilde{\varphi}_1(\tilde{x}_0)$, if $\tilde{Y}$ is a component in $\tilde{S}$ of the preimage of collars, then $\tilde{\varphi}_0(\tilde{Y}) = \tilde{\varphi}_1(\tilde{Y})$.

Now we use geometry: since $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$ are isometries, $\tilde{\varphi}_0|\tilde{Y} = \tilde{\varphi}_1|\tilde{Y}$.

Take the linear homotopy from $\tilde{\varphi}_0$ to $\tilde{\varphi}_1$. This homotopy descends to a homotopy $\phi_t : X_0 \to X_1$. \qed

The situation (described in Figure 3.16) is the following: we can foliate a collar with geodesic orthogonal to the boundary of the annulus. An arc can go around many times in the image of the collar. In the universal covering we pick a point and we have a horizontal segment. The image of this segment gives us a number $a - b$ called **twist coefficient**, which doesn’t depend on the point that I picked, on $\varphi$ and on the marking.
Figure 3.16: The red segment on universal covering of the annulus on the left side denote a preimage of the dashed segment, while the red segment in the right correspond to the image of the red segment. The number \((a - b)\) is the twist coefficient of the blue curve in the annulus.
If we consider \((X_0, f_0), (X_1, f_1) \in \mathcal{T}(S)\), the pants decomposition \(\varphi = \{\gamma_1, \ldots, \gamma_{3g-3}\}\) and we assume \(l_{\varphi}(X_0) = l_{\varphi}(X_1)\), then we have the twist coefficients
\[
\{\tau_i(X_0, X_1)\}_{i=1, \ldots, 3g-3}.
\]

**Lemma 3.7.8.** Under the above hypothesis we have:

1. If \(\tau_i(X_0, X_1)\) for every \(i\), then \(X_0 = X_1 \in \mathcal{T}(S)\);
2. If \(X_0, X_1, X_2 \in \mathcal{T}(S)\), then \(\tau_i(X_0, X_1) + \tau_i(X_1, X_2) = \tau_i(X_0, X_2)\), for every \(i\).

**Proof.** First we prove (1). Since \(\tau_i(X_0, X_1)\) for every \(i\), we can homotope \(\tilde{\phi}\) to be an isometry also in the collars. The homotopy descends to \(\phi_t : X_0 \to X_1\).

The proof of (2) follows by the observation that
\[
(a - b) + (b - c) = a - c,
\]
where \(a, b, c\) are the numbers in the Figure 3.17.

![Figure 3.17: Proof of (2) in the Lemma 3.7.8.](image)

**Theorem 3.7.9.** \(l^{-1}(a_1, \ldots, a_{3g-3}) \cong \mathbb{R}^{3g-3}\) with the affine structure (i.e., it is a vector space without 0).

Notice that all the twist coefficients are realized. We can construct an element in \(\mathcal{T}(S)\) with a certain twist coefficient \(\tau_i\) just cutting \(S\) along the \(i\)-th curve in the pants decomposition \(\varphi\) and gluing the curves that we got with a twist \(\tau_i\). Putting together all that we know till now we get
Theorem 3.7.10 (Fenchel-Nielsen Coordinates). \( \mathcal{F}(S) \) is an affine (trivial) bundle over \((\mathbb{R}^+)^{3g-3}\) with the fiber of dimension \(3g-3\):

\[
\mathbb{R}^{3g-3} \rightarrow \mathcal{F}(S) \rightarrow (\mathbb{R}^+)^{3g-3}.
\]

Corollary 3.7.11. \( \dim \mathcal{F}(S) = 6g - 6 \).

3.7.1 Dehn Twist

Definition 3.7.12. Let \( \gamma \) be an essential simple closed curve on \( S \) and \( A := S^1 \times [0, 2\pi] \) a collar neighborhood of \( \gamma \), where \( \gamma = S^1 \times \{ \pi \} \). We define the Dehn Twist \( D_\gamma \) as

\[
D_\gamma = \begin{cases} 
x, & \text{if } x \neq A \\
(\vartheta + t, t), & \text{if } x = (\vartheta, t) \in A.
\end{cases}
\]

![Figure 3.18: The Dehn Twist.](image)

Let \( \gamma \in \wp \) and consider \( (X, f) \in \mathcal{F}(S) \) and \( (X, f \circ D_\gamma) \). Notice that \( l_\varphi(X, f) = l_\varphi(X, f \circ D_\gamma) \), but the twist, called the Dehn twist of \( \gamma \), is

\[
\tau_\gamma((X, f), (X, f \circ D_\gamma)) = l_\gamma((X, f)) = l_\gamma((X, f \circ D_\gamma))
\]

Hence, \( D_\gamma \) is an example of a non-trivial twist.

Remark. Sometimes the twist coefficient is defined as our twist coefficient divided by the length of the curve \( l_\gamma \). With this notation the Dehn twist is 1.

3.7.2 Thick-Thin Decomposition

Theorem 3.7.13. Let \( X \) be a complete hyperbolic surface (without boundary unless explicitly stated). Then there exists an \( \epsilon > 0 \) such that every component of \( X^{<\epsilon} = \{ \text{set of points where injectivity radius is less than } \epsilon \} \) is either:

1. an annular neighborhood of a simple closed geodesic of length \(< 2\epsilon\); 
2. neighborhood of a puncture.
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Definition 3.7.14. The injectivity radius at a point $x \in X$ is
\[ \text{inj}(x) := \sup \{ r \mid D(x,r) \text{ is an embedded disk} \}. \]

Definition 3.7.15. (Second Definition of injectivity radius)
\[ \text{inj}(x) := \frac{1}{2} \inf \{ \text{length of all essential closed curves that contain } x \}. \]

The equivalence of the two above definitions can be easily seen when considering disks upstairs that project to non-disks downstairs (i.e., where a simple closed curve through $x$ exists).

Let $\gamma$ be a hyperbolic isometry, then $\mathbb{H}^2/\gamma \cong \text{annulus}$. Moreover, the injectivity radius is a constant function on equidistant curves from the core curve of the annulus.

Another fact is that the injectivity radius increases under covering space projections.

Fact 3.7.16. Let $\hat{\pi} : \hat{X} \to X$ be a covering space with $\hat{\pi}(\hat{x}) = x$, then $\text{inj}_{\hat{X}}(\hat{x}) \geq \text{inj}_X(x)$.

\[ X \]

\[ \gamma \]

Proof. \[ \square \]

Figure 3.20: Annular neighborhood of $\gamma$ will embed under covering map.

If $\gamma$ is a parabolic hyperbolic isometry then $\mathbb{H}^2/\gamma \cong \text{annulus}$. Since every parabolic is conjugate to the map $z \mapsto z + 1$ in the Upper Half Space model we see that horocycles are the simple closed curves on the cusp.
Figure 3.21: The horocycles are the simple closed curves on the cusp.

Recall that horocycles in $\mathbb{H}^2$ are circles tangent to $\partial \mathbb{H}^2$ (include horizontal lines in Upper Half Space model). In Euclidean space the curvature of a circle of radius $r$ is $\frac{1}{r}$.

In hyperbolic space the curvature of a circle of radius $r$ is $\coth(r)$, and the curvature of a curve lying at distance $d$ from a geodesic is $\tanh(d)$.

In order to prove the theorem it suffices to study the geometry of a pair of pants since there is a pants decomposition of the surface.

**Lemma 3.7.17.** (Collar Lemma) If $\gamma$ is a simple closed geodesic on $X$ then $\gamma$ has a collar neighborhood with

$$radius = \log \left| \frac{1 + \cosh \left( \frac{\lambda}{2} \right)}{\sinh \left( \frac{\lambda}{2} \right)} \right|,$$

where $\lambda = \text{length}(\gamma)$.

**Proof.** Extend $\gamma$ to a pants decomposition. Let $P$ be a pant with $\gamma \in \partial P$. We need to show that $\gamma$ has sufficiently large collar neighborhood on $P$. We are going to decompose $P$ into two ideal triangles.

The radius of the circle centered at $\frac{1 + e^{\lambda}}{2}$ is $\frac{\lambda - 1}{2}$ and thus $\sin(\theta) = \frac{\lambda - 1}{e^{\lambda} + 1} = \tanh \left( \frac{\lambda}{2} \right)$.

Consider the path $e^{it}$ for $\theta \leq t \leq \frac{\pi}{2}$, then the length of this path is

$$\int_{\theta}^{\frac{\pi}{2}} \frac{dt}{\sin(t)} = \log \left| \frac{1 + \cos(\theta)}{\sin(\theta)} \right| = \log \left| \frac{1 + \cosh \left( \frac{\lambda}{2} \right)}{\sinh \left( \frac{\lambda}{2} \right)} \right|.$$

**Question 3.7.18.** What is the length of the collar boundary?

**Answer 3.7.19.** Consider the retraction $re^{it} \mapsto ir$. Under this map vectors of length $\cosh(d)$ gets mapped to vectors of length 1. This implies that the length of the collar boundary is

$$\lambda \cosh \left[ \log \left( \frac{1 + \cosh \left( \frac{\lambda}{2} \right)}{\sinh \left( \frac{\lambda}{2} \right)} \right) \right].$$
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\[ z \mapsto e^\lambda z \]

geodesic being stabilized

Figure 3.22: Decomposition of \( P \) into two ideal triangles.

\[ \theta \]

0 \hspace{1cm} 1 \hspace{1cm} e^\lambda \hspace{1cm} \frac{1+e^\lambda}{2} \hspace{1cm} e^\lambda \]

geodesic being stabilized

Figure 3.23: circle centered at \( \frac{1+e^\lambda}{2} \) with radius \( \frac{e^\lambda - 1}{2} \).

Should also get the same answer by calculating the length of the path \( e^t e^{i\theta} \) for \( 0 \leq t \leq \lambda \).

Moreover, as \( \lambda \to 0 \) we see that

\[
\lambda \cosh \left[ \log \left( \frac{1 + \cosh(\frac{\lambda}{2})}{\sinh(\frac{\lambda}{2})} \right) \right] = \lambda \left( \frac{1 + \cosh(\frac{\lambda}{2})}{2 \sinh(\frac{\lambda}{2})} + \frac{\sinh(\frac{\lambda}{2})}{1 + \cosh(\frac{\lambda}{2})} \right) \to 2,
\]

when \( \lambda \to 0 \).

If we assume that \( X \) has a cusp, since the length of \( \lambda \) is 2 and geodesics are shorter than horocycles, the injectivity radius is less than 2.

\[ \square \]

3.7.3 Bers Constant

Let \( X \) be a finite area complete hyperbolic surface. Then there exists a constant \( B \) and a pants decomposition \( P \) depending only on \( \chi(X) \) such that the simple closed
3.8. $T(S)$ IS COMPLETE

We want to prove that $T(S)$ is complete. First we need the following proposition:

The disk that realizes the injectivity radius of a given point might or might not now intersect the boundary. If it does not then we can proceed as before. If the disk intersects the boundary we get three cases. The first is when the “disk” intersects the boundary in a point. Here we do not get a simple closed curve and hence choose another point. The second is when the “disk” intersects a boundary component in two points. Join them by a non-trivial arc. The regular neighborhood of the boundary union the arc is a pair of pants. The third case is when the “disk” intersects the boundary in one point in two different components. Here do same thing as in case 2: join the boundary components by a non-trivial arc. The regular neighborhood of the boundary components union the arc is a pair of pants.

3.8 $T(S)$ is complete

We want to prove that $T(S)$ is complete. First we need the following proposition:
Proposition 3.8.1. Let \((X, f), (Y, g)\) be points in \(\mathcal{T}(S)\) and \(\phi : X \to Y\) a \(K\)-quasi-conformal map such that \(\phi \circ f\) is homotopic to \(g\). Then for every essential simple closed curve \(\gamma\) in \(S\):

\[
\frac{1}{K} l_{g(\gamma)}(Y) \leq l_{f(\gamma)}(X) \leq K l_{g(\gamma)}(Y)
\]

where \(l_\alpha(X)\) denotes the hyperbolic length of the curve \(\alpha\) in the hyperbolic surface \(X\).

**Proof.** We lift \(X\) and \(Y\) to the universal coverings \(\tilde{X}\) and \(\tilde{Y}\) respectively and \(\phi : \tilde{X} \to \tilde{Y}\) to \(\tilde{\phi} : \tilde{X} = \mathbb{H}^2 \to \tilde{Y} = \mathbb{H}^2\). Let \(\hat{X}\) be the cover of \(X\) with \(\pi_1(\hat{X}) = < f(\gamma) >\) and \(\hat{Y}\) be the cover of \(Y\) with \(\pi_1(\hat{Y}) = < g(\gamma) >\).

The deck translation \(t\) corresponding to \(\gamma\) has the form \(z \to e^{i\lambda} z\) where \(\lambda = l_\gamma(X)\).

If \(A\) is the annulus in Figure 3.25 and \(R\) is the associated rectangle, then \(m(A) = m(R) = \frac{\pi}{2}\).

Hence,

\[
\frac{1}{K} l_{g(\gamma)}(Y) \leq l_{f(\gamma)}(X) \leq K l_{g(\gamma)}(Y)
\]

and so

\[
\frac{1}{K} l_{g(\gamma)}(Y) \leq l_{f(\gamma)}(X) \leq K l_{g(\gamma)}(Y).
\]

Using the geometric definition of a quasi-conformal map it is easy to prove the following

**Fact 3.8.2.** If \(\phi\) is \(K\)-quasi-conformal and \(\psi\) is \(K'\)-quasi-conformal then \(\phi \circ \psi\) is \(K \cdot K'\)-quasi-conformal.

Recall that

\[
d_T((X, f), (Y, g)) = \frac{1}{2} \log \inf \left\{ K \left| \exists \phi : X \to Y \text{ a } K\text{-quasi-conformal map such that } \phi \circ f \text{ is homotopic to } g \right. \right\}
\]

**Theorem 3.8.3.** \(\mathcal{T}(S)\) is a complete metric space.
Proof. Let \((X_i, f_i)\) be a Cauchy sequence. For all \(\epsilon > 0\) there exists \(N\) such that if \(i, j > N\) then there exists a \(K\)-quasi-conformal map \(\phi_{i,j} : X_i \to X_j\) (homotopic to \(f_j \circ f_i^{-1}\)) such that \(\frac{1}{2} \log K < \epsilon\).

Step 1: Let \(\{(X_i, f_i)\}_{i=1}^{\infty}\) be a sequence of points in \(\mathcal{T}(S)\) so that \(d_T(X_1, X_i) < \frac{1}{2} \log(J)\) (here we’re suppressing the markings). Let \(\phi_{1,i} : X_1 \to X_i\) be a \(J\)-quasi-conformal map in the right homotopy class. We show that there exists a Riemann surface \(X_{1,\infty}\) and a subsequence \(\phi_{i,k}\) that converges to a map \(\phi_{1,\infty} : X_1 \to X_{\infty}\). Moreover, there are maps \(\phi_{i,\infty} : X_i \to X_{\infty}\) and \(\phi_{j,\infty} \circ \phi_{i,j}\) is homotopic to \(\phi_{i,\infty}\). We want to apply the compactness theorem, but we can’t do so directly. We must lift the \(\phi_{1,i}\) to maps of the complex plane and must set things up so that three points are taken into compact sets. For each \(i, X_i\) is isometric to \(\mathbb{H}^2\). Fix a loop \(\gamma\) in \(S\) and let \(\gamma_i\) be the geodesic representative of \(f_i(\gamma)\) in \(X_i\). Pick a point \(p\) on \(\gamma_1\) and let \(p_i = \phi_{1,i}(p_1)\). Identify \(\tilde{X}_1\) with \(\mathbb{H}^2\) so that \(i\) projects to \(p_1\) and the \(y\)-axis projects to \(\gamma_1\). Identify \(\tilde{X}_i\) with \(\mathbb{H}^2\) so that the \(y\)-axis projects to \(\gamma_i\). Define \(\tilde{\phi}_{1,j}\) to be the lift of \(\phi_{1,j}\) which takes \(i\) to the lift of \(p_j\) which is closest to \(i \in \mathbb{H}^2\).

The deck translation \(t_j\) corresponding to \(\gamma_j\) has the form \(z \to e^{i\lambda_j}z\) where \(\lambda_j = b_{\gamma_j}(X_j)\). By Proposition 3.8.1, \(\frac{1}{J}\lambda_1 \leq \lambda_j < J\lambda_1\). Thus \(0 < \lambda_{\min} \leq \lambda_j \leq \lambda_{\max}\). Define \(M(s,t) = \{re^{i\theta} | s \leq r \leq t, 0 < \theta < \pi\}\). Then \(M = M(\frac{1}{\sqrt{\lambda_{\min}}}, \sqrt{\lambda_{\max}})\) is a fundamental domain for \(t_1\) containing \(i\). \(\tilde{\phi}_{1,j}(M) \subseteq M(\frac{1}{\sqrt{\lambda_{\max}}}, \sqrt{\lambda_{\max}})\). In particular, \(\tilde{\phi}_{1,j}\) takes \(i\) into a compact subset of \(\mathbb{C}\). In addition, since lifts commute with the action of the deck translations, \(\phi_{1,j} \circ t_{\gamma_j} = t_{\gamma_j} \circ \phi_{1,j}\). Thus, \(\tilde{\phi}_{1,j}(t^n_{\gamma_j}(i)) \in t^n_{\gamma_j} \circ \tilde{\phi}_{1,j}(i)\). So \(\lambda_{\min} \frac{1}{\sqrt{\lambda_{\max}}} \leq \tilde{\phi}_{1,j}(t^n_{\gamma_j}(i)) \leq \lambda_{\max} \sqrt{\lambda_{\max}}\). Thus \(\tilde{\phi}_{1,j}(t^n_{\gamma_j}(i)) \in M(\lambda_{\min} \frac{1}{\sqrt{\lambda_{\max}}}, \lambda_{\max} \sqrt{\lambda_{\max}})\). For \(n\) large enough \(\lambda_{\max}^{-n} \sqrt{\lambda_{\max}} < \frac{1}{\lambda_{\min}}\) and \(\sqrt{\lambda_{\max}} < \lambda_{\min}^{-n} \sqrt{\lambda_{\max}}\) making the three sets \(M(\lambda_{\max}^{-n} \frac{1}{\sqrt{\lambda_{\max}}}, \lambda_{\max}^{-n} \sqrt{\lambda_{\max}}), M(\frac{1}{\sqrt{\lambda_{\max}}}, \sqrt{\lambda_{\max}}), M(\lambda_{\min}^{-n} \frac{1}{\sqrt{\lambda_{\max}}}, \lambda_{\max}^{-n} \sqrt{\lambda_{\max}})\) disjoint (see Figure 3.26). Now, we may apply the compactness theorem to get a subsequence \(j_k\) and a map \(\phi_{1,\infty}\) so that \(\phi_{1,j_k} \to \phi_{1,\infty}\) and the limit map is a \(J\)-quasi-conformal homeomorphism from the upper half plane to itself.

Figure 3.26: We constructs the lifts \(\tilde{\phi}_{1,j}\) so that three points are taken to compact sets.
Now, for each \( j \) we have an injective homomorphism \( \rho_j : \pi_1(S) \to \text{Aut}(\mathbb{H}^2) \). We have the following commutative diagrams:

\[
\begin{array}{c c}
\tilde{X}_1 \cong \mathbb{H}^2 & \xrightarrow{\phi_{1,j}} \tilde{X}_j \cong \mathbb{H}^2 \\
\downarrow \rho_1(g) & \downarrow \rho_j(g) \\
X_1 & \xrightarrow{\phi_{1,j}} X_j.
\end{array}
\]

Therefore, define \( \rho_\infty : \pi_1(S) \to \text{Aut}(\mathbb{H}^2) \) by \( \rho_\infty(g) = \tilde{\phi}_{1,\infty} \circ \rho_1 \circ \tilde{\phi}_{1,\infty}^{-1} \). Since \( \rho_\infty \) is discrete and faithful, we can define \( X_{1,\infty} := \mathbb{H}^2/\rho_\infty(\pi_1(S)) \). We get a point \( (X_{1,\infty}, \phi_{1,\infty} \circ f_1) \in T(S) \).

Step 2: Notice that in our previous construction the \( \rho_1 \) action was topologically conjugate to both \( \rho_2 \) and \( \rho_\infty \). Therefore, \( \rho_2 \) and \( \rho_\infty \) are conjugate through the map \( \tilde{\phi}_{1,2} \circ \tilde{\phi}_{1,\infty} \). Thus \( X_{2,\infty} = X_{1,\infty} \) and we denote this limit by \( X_\infty \).

In conclusion (by passing to a subsequence), we get a sequence of maps \( \phi_{i,\infty} : X_i \to X_\infty \) such that \( \phi_{i,\infty} \circ f_i \sim \phi_{j,\infty} \circ f_j \) and where \( \phi_{i,\infty} \) is \((1 + \epsilon)\)-quasi-conformal. This concludes the proof that \( T(S) \) is complete. \( \square \)

### 3.9 Constructing Riemann Surfaces

There are many ways to construct Riemann Surfaces:

- via Hyperbolic Geometry: glueing pair of pants;
- we can construct a Riemann surface \( X \) using the universal cover \( \mathbb{H}^2 : \mathbb{H}^2/\Gamma = X \);
- using polynomials in \( \mathbb{C}^2 \) (algebraic geometry);
- surfaces as quotient spaces.

We will think of surfaces as quotient spaces. Let \( P \) be a polygon in \( \mathbb{C} \) and identify pairs of sides with orientation-preserving Euclidean isometries. Then this determines a Riemann surface structure since the link of a vertex is a compact 1-manifold and hence \( S^1 \).

**Definition 3.9.1.** A *translation surface* \( X \) is a Riemann Surface such that for any two charts \( (U, \phi) \) and \( (V, \psi) \) the transition map

\[
\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)
\]

is a translation.

It is clear that both on the inside of the polygon \( P \) and on the edges that the transition maps are Euclidean isometries. Around the vertices define the chart by mapping a neighborhood of the vertex by the map \( z \mapsto z^{2\pi} \) where \( \alpha \) is the sum of
the angles around the vertex. We also want to require that the Euclidean isometries that we are using are pure translations. One way to visualize such maps is by using subdivided rectangles.

Let $T(\mathbb{C})$ be the group of translations of $\mathbb{C}$ and $X = (P, \mathcal{G})$ where $P$ is our polygon and $\mathcal{G} \subset T(\mathbb{C})$ are the gluings. Let $\phi_K(x+iy) = Kx+iy$. Then $X_K = (\phi_K(P), \phi_K(\mathcal{G}))$ defines a new polygon with gluings still in $T(\mathbb{C})$ (the map $\phi_K$ stretches in the $x$-direction which does not affect the gluing map). This map $\phi_K$ descends to a map between the quotient Riemann surfaces.

**Theorem 3.9.2** (Teichmüller’s Theorem). Let $X$ and $X_K$ as above. Then

$$d_T(X, X_K) = \frac{1}{2 \log K}$$

and if $\phi : X \to X_K$ is $K$-quasi-conformal then $\phi = \phi_K$.

In other words, the Teichmüller Theorem says that if $\psi : X \to X_K$, $\psi \sim \phi_K$ and $\psi$ is $K'$-quasi-conformal, then $K' \geq K$ and if $K' = K$, then $\psi = \phi_K$.

**Definition 3.9.3.** $\phi_K$ is called the **Teichmüller map**.

The proof of this theorem is at its heart the same as the proof of Grötzsch’s theorem. In order to get a bijection between the space of rectangles and the Teichmüller space of a surface, we allow our gluings to live in $T^+(\mathbb{C})$ which is all translations and rotations by $\pi$. 
3.10 1-forms

Let $\omega$ be a complex 1-form on $S$ and $\gamma : [a, b] \to S$ a path. Then we can integrate the
1-form over this path via

$$\int_{\gamma} \omega = \int_{a}^{b} \omega_{\gamma(t)} \gamma'(t) dt$$

at each point $x \in S$ where $\omega_x : T_x S \to \mathbb{C}$.

In $\mathbb{R}^2$ the two standard 1-forms are $dx(u, v) = u$ and $dy(u, v) = v$. Every 1-form
on $\Omega \subset \mathbb{R}^2$ is of the form $f dx + g dy$ for $f, g : \Omega \to \mathbb{C}$. In $\mathbb{C}$ we have the following
1-forms:

$$
\begin{align*}
    dz &= dx + idy \\
    d\bar{z} &= dx - idy \\
    dx &= \frac{1}{2} (dz + d\bar{z}) \\
    dy &= \frac{1}{2} (dz - d\bar{z})
\end{align*}
$$

$$
    f dx + g dy = \frac{1}{2} (f - ig) dz + \frac{1}{2} (f + ig) d\bar{z}.
$$

Hence, every complex value 1-form on $\Omega$ is of the form

$$
    \frac{1}{2} (f - ig) dz + \frac{1}{2} (f + ig) d\bar{z}.
$$

Now, for $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) we have

$$
    df = f_x dx + f_y dy = \frac{1}{2} (f_x - if_y) dz + \frac{1}{2} (f_x + if_y) d\bar{z}.
$$

**Definition 3.10.1.** Let $f : \Omega \to \mathbb{C}$ be a map. Then

$$
    f_z = \frac{1}{2} (f_x - if_y) \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2} (f_x + if_y).
$$

The Cauchy-Riemann equations say that $f$ is holomorphic if and only if $f_z = 0$. If we are going to change coordinates, then for $\phi(z) = w$ and $\alpha = \alpha(w) dw$ we have

$$
    \phi^* \alpha_z(u, v) = \alpha_{\phi(z)} \phi_z(u, v).
$$

If $\phi = (\phi_1, \phi_2)$, then the Jacobian is

$$
\begin{pmatrix}
    (\phi_1)_x & (\phi_1)_y \\
    (\phi_2)_x & (\phi_2)_y
\end{pmatrix}
\begin{pmatrix}
    u \\
    v
\end{pmatrix}.
$$

If $\phi$ is holomorphic, then $(\phi_z)_z(u, v) = \phi'(z)(u + iv)$ and $\phi^* \alpha = \phi'(z) \alpha(\phi(z)) dz$.

Analogously, if $\beta = \beta(w) d\bar{w}$, then $\phi^* \beta = \phi'(z) \beta(\phi(z)) d\bar{z}$.

Therefore, if $\omega$ is a $\mathbb{C}$-valued 1-form on a Riemann surface, there is a canonical
decomposition into holomorphic and anti-holomorphic parts. We have seen that the
tangent bundle of $X$ tensored with $\mathbb{C}$ as a decomposition in holomorphic and anti-
holomorphic tangent bundle.
Let $\alpha$ be a section of the holomorphic tangent bundle then it “only has $dz$ part in local coordinates”. In local coordinates, $\alpha = \alpha(z)dz$ and $d\alpha = 0 \iff d\alpha \wedge dz \iff \alpha_zdz \wedge dz + \alpha_zdz \wedge dz \iff \alpha_z = 0$. Thus, $\alpha$ is holomorphic if and only if it is closed. Now, let $\alpha$ be a holomorphic 1-form. Then the integral $\int_\gamma \alpha$ with $\gamma(0) = z_0, \gamma(1) = z$ is well defined since $\alpha$ is closed. If we are not at a zero of the 1-form, then this defines a chart.

Let $\alpha$ be a holomorphic 1-form on $X$. In local coordinate, $\alpha = \alpha_U,\phi dz$ with $\alpha_U,\phi$ a holomorphic function. Let $p \in X$ and $(U, \phi)$ be a chart with $p \in U$. If $\alpha_U,\phi(\phi(p)) = 0$, then for all charts $(V, \psi)$ with $p \in V \alpha_{V,\psi}(\psi(p)) = 0$ since $(\psi \circ \phi^{-1})'(z) \neq 0$ and

$$\alpha_{U,\phi}(z) = \alpha_{V,\psi}(\psi \circ \phi^{-1}(z))(\psi \circ \phi^{-1})'(z).$$

Hence, $\alpha(p) = 0$ is well-defined, the order of a zero is well-defined and moreover, the zero’s are isolated.

Let $\alpha_U,\phi dz$ represent $\alpha$ on a chart where $\alpha \neq 0$. For any $p \in U$ there exists $\beta$ defined on a neighborhood at $\phi(p)$ such that $\beta' = \alpha_U,\phi$.

Since $\beta'(\phi(p)) \neq 0$, $\beta$ is a local homeomorphism. Shrinking the chart (if necessary), $(U, \beta \circ \phi)$ is a new chart such that $\alpha = \alpha_{U,\beta \circ \phi}dz$. What is $\alpha_{U,\beta \circ \phi}$?

Since

$$\alpha_{U,\phi}(z)dz = \alpha_{U,\beta \circ \phi}(\beta(z)) \cdot \beta'(z)dz = \alpha_{U,\beta \circ \phi}(\beta(z))\alpha_{U,\phi}(z)dz,$$

$\alpha_{U,\beta \circ \phi} = 1$.

Hence, if $p \in X$ and $\alpha(p) \neq 0$, then there exists a chart $(U, \phi)$ with $p \in U$ such that $\alpha = dz$ on $U$. Let $A_\alpha$ be the set of charts with $\alpha = dz$ and let $B_\alpha$ be the set of charts $(U, \phi)$ such that $(V, \psi) \in A_\alpha$ and $\phi = \phi_K \circ \psi$.

Since $\alpha_{U,\phi}(z) = \alpha_{U,\beta \circ \phi}(\beta(z)) \cdot \beta'(z)$, $\beta'(z) = 1$ and so $\beta(z) = z + \text{const}$.

Hence, $B_\alpha$ defines a new Riemann translation surface on the surface $S$.

In conclusion, $A_\alpha$ determines a Riemann surface $X$, $B_\alpha$ determines a Riemann surface $X_K$ and $\phi_K : X \to X_K$. We are in the same situation as in the case of the polygons. What does it happen in the zero’s? If $\alpha(p) = 0$, then there exists a chart such that $\alpha(z) = z^ndz$, $n \geq 1$. Indeed, take the map $z \mapsto z^n$ and pullback the 1-form $dz$. We get $nz^{n-1}dz$. $\alpha$ is the pull-back of $dz$ using the map $z \mapsto z^n$.

How do we get the rotation by $\pi$? To find an answer to this question we need to talk about the quadratic differentials.
Chapter 4
Quadratic Differentials

We will define quadratic differentials and in particular, holomorphic quadratic differentials. Moreover, we will prove that the set of holomorphic quadratic differentials with area less than 1 is homeomorphic to $T(S)$.

4.1 Holomorphic quadratic differentials

Definition 4.1.1. A quadratic differential $\Phi$ is a collection of functions $\{\phi_U\}$ on local charts such that
\[ \phi_U(z) = \phi_V(f(z))f'(z), \]
where $f$ is the transition map.

A holomorphic quadratic differential is a quadratic differential $\Phi$ such that the functions $\{\phi_U\}$ are holomorphic.

If $\phi_U(z)$ in $U$, $\phi_V(z)$ in $V$, $U \cap V \neq \emptyset$ and $f(z) = w$ is the (holomorphic) transition map, then
\[ \phi_U(z)dz^2 = \phi_V(f(z))f'(z)^2dz^2. \]

Definition 4.1.2. Let $p, q \in \mathbb{Z}$. A $(p,q)$-differential $\Phi$ is a collection of functions $\{\phi_U\}$ on local charts such that if $\phi_U(z)dz^pdz^q$ in $U$ and $\phi_V(z)dw^pd\bar{w}^q$ in $V$, then
\[ \phi_U(z)dz^pdz^q = \phi_V(f(z))f'(z)^p\overline{f'(z)}^qdw^pd\bar{w}^q, \]
where $f$ is the (holomorphic) transition map.

If $\phi_U$’s are holomorphic, then $\Phi$ is called holomorphic $(p,q)$-differential.

The cases that we are interested in are $(1,0)$, $(2,0)$, $(-1,1)$ (Beltrami differentials), $(1,1)$ (2-forms), $(0,0)$ (functions).

Remark. The set of $(p,q)$-differentials form a $\mathbb{C}$-vector space.

Let $Q(X)$ denote the set of holomorphic quadratic differentials in $X$.

In the holomorphic setting we can apply the Riemann-Rock Theorem and we have:

Theorem 4.1.3. $\dim Q(X) = \dim(T(S))$. 

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Quadratic differentials behave very nicely. Indeed, they satisfy the following properties:

1. If $\Phi$ is a $(p, q)$-differential, then $\overline{\Phi}$ is a $(q, p)$-differential.

2. If $\Phi$ is a $(p, q)$-differential and $\Psi$ is a $(p', q')$-differential, then $\Phi \cdot \Psi$ is a $(p + p', q + q')$-differential.

3. If $\Phi$ is a $(2, 0)$-differential, then $|\Phi| = \sqrt{\Phi \cdot \overline{\Phi}}$ is a $(1, 1)$-differential. Hence, $Area = \int_X |\Phi|$ is defined since $dzd\bar{z} = (dx + idy)(dx - idy) = -2idxdy$,

$$\phi_U(z)dzd\bar{z} = \phi_V(f(z))f'(z)f'(z)dzd\bar{z} = \phi_V(f(z))|f'(z)|^2dzd\bar{z},$$

and so

$$\int \phi_U dzd\bar{z} = \int -2i\Phi dxdy = \int \phi_V dwd\bar{w}.$$

4. If $\alpha$ is a 1-form, then $Area = \int_X |\alpha|^2$.

If $\alpha$ is a holomorphic 1-form, then we defined a chart at $p \in X$ with $\alpha(p) \neq 0$ by $\int_\gamma \alpha$, where $\gamma$ is a path from $p$ to $q$. Since $\alpha$ is closed, the integral $\int_\gamma \alpha$ is well defined.

If $\Phi$ is a quadratic differential where $\phi_U$ represent $\Phi$ in a chart $U$ about $p$ and $p \in X$ with $\Phi(p) \neq 0$, then there exists $\alpha_U$ such that $\alpha^2_U = \phi_U$ and we can define a chart taking $\int_\gamma \sqrt{\Phi}$, where $\gamma$ is a path from $p$ to $q$. This gives a family of charts. Indeed, there exists a (transition) map $\beta$ such that $\beta' = \alpha_U$. If $(U, \psi)$ was the original chart containing $p$, then $(U, \beta \circ \psi)$ is the new chart (possibly after shrinking $U$ since we want a simply-connected domain) and on this chart $\Phi$ is $\overline{\phi_U}dz^2$. What is $\overline{\phi_U}$?

Since

$$\phi_U(z)dz^2 = \overline{\phi_U}(\beta(z))\beta'(z)dz^2 = \overline{\phi_U}(\beta(z))\alpha^2_U(z)dz^2 = \overline{\phi_U}(\beta(z))\phi_U(z)dz^2,$$

$\overline{\phi_U} = 1$.

Let $\mathcal{A}_\Phi$ be the set of charts with transition maps given by translations and rotations by $\pi$ (we ignore the zero’s for the moment). In charts, $\Phi = dz^2$. Let $\mathcal{B}_\Phi$ be the set of charts $(U, \phi)$ such that $(V, \psi) \in \mathcal{A}_\Phi$ and $\phi = \phi_K \circ \psi$.

If $(U, \psi_0)$ and $(V, \psi_1)$ are charts such that $\Phi = \phi_U dz^2$ on $(U, \psi_0)$ and $\Phi = \phi_V dz^2$ on $(V, \psi_1)$ with $\phi_U = \phi_V = 1$, then the derivative of the transition map $(\psi_1 \circ \psi_0^{-1})' = \pm 1$.

So $\psi_1 \circ \psi_0^{-1}(z) = \pm z + const$, that is the transition maps are given by translations and rotations by $\pi$.

A consequence of the Teichmüller Theorem is the following.

Let $Q_1(X) := \{ \Phi \in Q(X) | area(\Phi) < 1 \}$. We define $T_X : Q_1(X) \rightarrow T(S)$ in the following way: if $\Phi \in Q_1(X)$, let $k := area(\Phi)$ and $K := \frac{1 + k}{1 - k}$. The quadratic differential $\Phi$ determines $\mathcal{A}_\Phi$ and so $\mathcal{B}_\Phi$. This gives a Riemann surface $Y$ and a map $f_\Phi : X \rightarrow Y$. Let $T_X(\Phi) = (Y, f_\Phi)$. $T_X$ is obviously continuous.

By the Teichmüller Theorem, $T_X$ is an injective and proper map since $d_T(X, T_X(\Phi)) = \frac{1}{2}log(K)$. 
Hence, $T_X$ is a proper, injective, continuous map between two copies of $\mathbb{R}^n$ of the same dimension (since $Q_1(X)$ is open in $Q(X)$ and $\dim Q(X) = \dim(T(S))$). By the Invariance of Domain, $T_X$ is a homeomorphism.

**Remark.** If $\Phi$ is fixed, then $\lambda \mapsto T_X(s(\lambda)\Phi)$ is a geodesic, where $s(\lambda)$ is a function in $\lambda$. Moreover, there exists a unique geodesic between any two points in $T(S)$.

**Example 4.1.4.** Let $\lambda \in \Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ define a map from the complex plain to itself by $\phi^\lambda(z) = z + \lambda \overline{z}$. The Beltrami differential in this case is $\mu(\phi^\lambda) = \frac{(\phi^\lambda)\overline{z}}{(\phi^\lambda)'z} = \lambda$.

Let $P$ be the polygon in Figure ?? and denote $P^\lambda = \phi^\lambda(P)$. The gluing maps are $g_j(z) = z + c_j$, where $c_1 = 2, c_2 = 1, c_3 = 2i, c_4 = i$. We define the gluing on $P^\lambda$ so that the following diagram commutes:

$$
\begin{array}{ccc}
P & \xrightarrow{\phi^\lambda} & P^\lambda \\
\downarrow{g_i} & & \downarrow{g_i^\lambda} \\
P & \xrightarrow{\phi^\lambda} & P^\lambda.
\end{array}
$$

Let $X^\lambda$ be the Riemann surface defined by quotienting $P^\lambda$ by the gluing $g_i^\lambda$. The map $\phi^\lambda : P \to P^\lambda$ descends to a map $f^\lambda : X_0 \to X_\lambda$. Using Teichmüller’s Theorem, we get $d_T(X_0, X^\lambda) = \log \left(\frac{1+|\lambda|}{1-|\lambda|}\right) = d_\Delta(0, \lambda)$ where $d_\Delta(\cdot, \cdot)$ denotes the distance from 0 to $\lambda$ in the disc model of the hyperbolic plane. Similarly, for each $\lambda, \lambda' \in \Delta$ define $f^{\lambda, \lambda'} = f^{\lambda'} \circ (f^\lambda)^{-1} : X^\lambda \to X^{\lambda'}$. Again by Teichmüller’s Theorem, we get that $d_T(X^\lambda, X^{\lambda'}) = d_\Delta(\lambda, \lambda')$. Therefore, the map $\lambda \mapsto X^\lambda$ defines an isometric embedding from $\Delta$ to $T(S)$.

Now we prove the Teichmüller’s Theorem.

**Proof of Teichmüller’s Theorem.** Let $f : X \to X^\lambda$ be the Teichmüller map as previously defined and let $K$ be its quasi-conformal constant. Let $g : X \to X^\lambda$ be any other $L$-quasi-conformal map from $X$ to $X^\lambda$. We wish to prove that $L \geq K$ with equality iff $g = f$.

Let $\alpha(p)$ be the direction of maximal stretch for $f$ at $p$. Define $\lambda(g, p) = |\partial_{\alpha(p)} g|$ where $\partial_{\alpha(p)} g$ is the directional derivative of $g$ in the $\alpha$ direction. For notational simplicity, assume that $f$ is defined in a way where in every chart $\alpha(\phi(p))$ is the $x$-direction.

Following the method of proof of Grötzsch’s theorem, in local coordinates we have:

$$
|g|^2 \leq (|g_z| + |g_\bar{z}|)^2 = \frac{|g_z| + |g_\bar{z}|}{|g_z| - |g_\bar{z}|}(|g_z|^2 - |g_\bar{z}|^2) \leq LJ_p(g),
$$

where $J_p(g)$ is the Jacobian of $g$ at $p$. Therefore, we have:

$$
\lambda(g, p)^2 \leq LJ_p(g).
$$

And by invoking Schwartz’s inequality we get:

1 exists almost everywhere?

2 Now $f$ is not given in terms of the map $\phi^\lambda$. 

\[(f_X \lambda(g, p))^2 \leq \int_X \lambda(g, p)^2 \leq \int_X \lambda(g, p)^2 f_X 1 \leq \int_X L \lambda(p) dx \cdot \text{Area}(X) = L \text{Area}(Y) \text{Area}(X).\]

Note that the area of $X$ and $Y$ are calculated with respect to the volume form which corresponds to the vectors of maximal/minimal stretch of $f^3$. Now, if we also had $\text{Area}(Y) \leq \int_X \lambda(g, p)$, we would get $L \leq \frac{\text{Area}(Y)}{\text{Area}(X)} = K$. We need the following lemma.

**Lemma 4.1.5.** Let $(X, \rho)$ be a compact metric space, and $f : X \rightarrow X$ a homeomorphism homotopic to the identity. There exists a constant $M$ such that for every geodesic segment $\gamma$, $|f(\gamma)|_\rho > |\gamma|_\rho - 2M$, where the constant $M$ only depends on $f$.

This Lemma says that the image of a geodesic will not get too short.

**Proof.** We lift $f$ to the universal cover $\tilde{X}$: $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$, where $X = \tilde{X}/\pi_1(X)$. We can choose $\tilde{f}$ such that $\tilde{f} \circ \alpha = \alpha \circ \tilde{f}$ for all $\alpha \in \pi_1(X)$. By compactness, there exists $M$ such that $d_\rho(\tilde{f}(\tilde{x}), \tilde{x}) \leq M$, for all $\tilde{x} \in \tilde{X}$. Then, since $\alpha \in \pi_1(X)$ is a Deck transformation, we have

\[
d_\rho(\tilde{f}(\tilde{x}), \tilde{x}) = d_\rho(\alpha \circ \tilde{f}(\tilde{x}), \alpha(\tilde{x})) = d_\rho(\tilde{f}(\alpha(\tilde{x})), \alpha(\tilde{x})),
\]

for all $\alpha \in \pi_1(X)$.

If $x \in X$, define $\psi(x) = d_\rho(\tilde{f}(\tilde{x}), \tilde{x})$, where $\tilde{x}$ is the preimage of $x$. The function $\psi$ is continuous on $X$, so there exists $M$ such that $\psi(x) \leq M$ for all $x \in X$. Hence, if $\gamma$ is a geodesic from $x_0$ to $x_1$, and respectively, $\tilde{x}_0$ and $\tilde{x}_1$ are the preimages, then

\[
d_\rho(\tilde{x}_0, \tilde{x}_1) \leq d_\rho(\tilde{f}(\tilde{x}_0), \tilde{x}_0) + d_\rho(\tilde{f}(\tilde{x}_0), \tilde{f}(\tilde{x}_1)) + d_\rho(\tilde{f}(\tilde{x}_1), \tilde{x}_1) \leq 2M + d_\rho(\tilde{f}(\tilde{x}_0), \tilde{f}(\tilde{x}_1))
\]

and projecting this inequality to $X$, we get

\[|\gamma|_\rho \leq 2M + |f(\gamma)|_\rho.\]

Consider the map $h = g \circ f^{-1} : Y \rightarrow Y$. Define

\[\lambda(h, q) := |g_*(q)|.\]

Since $f^{-1}$ shrinks by $K$ in the $x$-direction, we get $\lambda(h, f(p)) = \frac{1}{K} \lambda(g, p)$ which implies

\[
\int_X \lambda(g, p) = \int_X K \lambda(h, f(p)) dX = \int_Y K \lambda(h, q) J_p(f) dY = \int_Y \lambda(h, q). \tag{4.3}
\]

We must show that $\int_Y \lambda(h, q) \geq \text{Area}(Y)$.

**Proposition 4.1.6.** If $h : Y \rightarrow Y$, $h \sim id$, then $\int_Y \lambda(h, q) dA_Y \geq \text{Area}(Y)$.

\[\text{Huh?}\]
4.2. BRANCHED DOUBLE COVERS

Proof. First, we need to do the setup.
We consider the geodesic flow on \( Y \) in the \( x \)-direction. This flow is well defined since we have a well defined \( x \)-direction and \( y \)-direction. This flow defines a map \( \varphi_t : Y \to Y \) in the following way: if \((U, \phi)\) is a chart, \( p \in U \), for small \(|t| < \varepsilon\) define \( \varphi_t(p) := \phi^{-1}(\phi(p) + t) \).

We are working with the singular Euclidean metric and so the flow is not defined in the singularities. If the flow is defined, then \( \varphi_t \circ \varphi_{t_1} = \varphi_{t_0 + t_1} \).

Let \( \Omega_t := \{p \in Y | \varphi_t(p) \text{ is defined}\} \subset Y \) and let \( \Omega := \bigcap_{t \in \mathbb{R}} \Omega_t \). We have \( \text{area}(\Omega_t) = \text{area}(Y) \) and in general \( Y \setminus \Omega \) will be dense. For example, think about an \( x \)-direction with irrational slope in the torus.

With this setup we can prove the theorem. First, since \( \varphi_t \) is an isometry we have

\[
\int_{-L}^{L} \int_{\Omega_t} \lambda(h, \varphi_t(p))dA_Y dt = \int_{-L}^{L} \left( \int_{\Omega_t} \lambda(h, p)dA_Y \right) dt = 2L \int_Y \lambda(h, p)dA_Y \tag{4.4}
\]

and also, by Lemma 4.1.5,

\[
\int_{-L}^{L} \lambda(h, \varphi_t(p)) dt \geq 2L - 2M. \tag{4.5}
\]

The equality 4.4 and the disinequality 4.1.5 imply:

\[
2L \int_Y \lambda(h, p)dA_Y = \int_{-L}^{L} \int_{\Omega_t} \lambda(h, \varphi_t(p))dA_Y dt =
\]

\[
= \int_{\Omega_t} \int_{-L}^{L} \lambda(h, \varphi_t(p)) dtdA_Y \geq \int_{\Omega_t} (2L - 2M)dA_y = 2(L - M)\text{area}(Y).
\]

Hence,

\[
\int_Y \lambda(h, p)dA_Y \geq \frac{2(L - M)}{2L} \text{area}(Y) \to \text{area}(Y)
\]
as \( L \to \infty. \)

This concludes the proof of the Teichmüller’s Theorem.

4.2 Branched Double Covers

Let \( X = (P, \mathcal{G}) \). We want to build a double cover of \( X \).

Let \( P' = r(P) \), where \( r(z) = -z + c \) is a rotation.

Define \( \mathcal{G}' \) in the following way:

1. If \( g \in \mathcal{G} \) and \( g(z) = z + c \), then \( g \in \mathcal{G}' \) and \( r \circ g \circ r^{-1} \in \mathcal{G}' \);

2. If \( g \in \mathcal{G} \) and \( g(z) = -z + c \), then \( r \circ g \in \mathcal{G}' \) and \( g \circ r^{-1} \in \mathcal{G}' \).
So we have $X' = (P \sqcup P', \mathcal{G}')$ and a map $\pi : X' \to X$ such that

$$
\begin{array}{c}
X' \\
\pi \\
\downarrow \quad \downarrow \hspace{1cm} h \circ \pi \\
X \\
\end{array}
$$

We can apply the same argument in the proof of the Teichmüller’s Theorem with $h \circ \pi$ instead of $h$.

**Theorem 4.2.1** (Teichmüller Existence and Unique Theorem). Let $X, Y \in \mathcal{T}(S)$. There exists a unique map $f : X \to Y$ such that $f$ is $K$-quasi-conformal and $d_T(X,Y) = \frac{1}{2} \log(K)$.

**Proof.** Define $T_X : Q_1(X) \to \mathcal{T}(S)$ in the following way. Given $\Phi \in Q_1(X)$, let $f_\Phi : X \to Y_\Phi$ be the Teichmüller map such that

$$
\frac{(f_\Phi)_z}{(f_\Phi)_\bar{z}} = \text{area}(\Phi).
$$

We have the following

1. $T_X$ is continuous;
2. $\dim \mathbb{R}(Q_1(X)) = 6g - 6$ (Riemann-Roch Theorem);
3. $\dim \mathbb{R}(\mathcal{T}(S)) = 6g - 6$ (Fenchel-Nielsen coordinates);
4. $T_x$ is injective (the Teichmüller map is unique);
5. $T_X$ is proper (if $\Phi_i$ is such that $\text{area}(\Phi_i) \to 1$, then $d_T(X, T_X(\Phi_i)) \to \infty$).

By the Invariance of Domain, $T_X$ is a homeomorphism.

Let $\Phi \in Q(X)$ with $\text{area}(\Phi) = 1$. Define

$$
T_\Phi : \mathbb{H}^2 \equiv \Delta \to \mathcal{T}(S),
$$

$T_\Phi(\lambda) := T_X(\lambda \Phi)$.

We have shown that $T_\Phi$ is an isometry between the hyperbolic disc and $\mathcal{T}(S)$ (i.e., $\Delta$ is a totally geodesic subspace of $\mathcal{T}(S)$) and so if $X, Y \in \mathcal{T}(S)$, there exists an isometric embedding $\mathbb{H}^2 \to \mathcal{T}(S)$ such that $X$ and $Y$ are in the image, then there is a geodesic connecting them, because there is a geodesic connecting their preimage.

**Remark.** Here we think $\mathbb{H}^2$ as a complex manifold.

Our next goal is to show that the Teichmüller space is not negatively curved and is not $\text{CAT}(0)$.

First, we need to introduce the Strebel differentials.
4.2. BRANCHED DOUBLE COVERS

4.2.1 Strebel Differentials

Glue the opposite sides of the polygons in Figure 4.1 to get a Quadratic Differential on $S_2$. Notice that horizontal lines close up to simple closed curves. Up to homotopy, there are respectively two and one homotopy classes of horizontal trajectories going from left to right.

Definition 4.2.2. Let $\Phi \in Q(X)$, $\Phi$ is a Strebel differential if all but finitely many (nonsingular) horizontal trajectories are simple closed curves.

An $\epsilon$-neighborhood of a closed trajectory is an annulus foliated by closed trajectories. The number of possible annuli is $3g-3$. A Strebel differential decomposes $S$ into collection of cylinders representing homotopy classes $\gamma_1, \ldots, \gamma_k$. These cylinders are standard Euclidean cylinders. An example of a non-Strebel differential can be found by rotating any Strebel differential by some complex number that is not a multiple of $\frac{\pi}{2}$. In addition, there are quadratic differentials that when multiplied by any complex number are not Strebel.

Theorem 4.2.3 (Strebel). Given $X$, a collection of disjoint simple closed curves $\gamma_1, \ldots, \gamma_k$ and heights $h_1, \ldots, h_k$, there exists a unique Strebel differential $\Phi$ on $X$ with cylinders $\gamma_1, \ldots, \gamma_k$ and heights $h_1, \ldots, h_k$.

Recall that given any conformal annulus $A$ with $0 < m(A) < \infty$, there exists a Euclidean metric on $A$ with height $h$.

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_k\}$ be a collection of disjoint open annuli on $X$ with core curves $\gamma_1, \ldots, \gamma_k$. Define

$$m(\mathcal{A}) := \sum \frac{h_i^2}{m(A_i)}.$$

Step 1: There exists an $\mathcal{A}$ that realizes $inf\{m(A_j)\}$, where the infimum is over all the possible collection of disjoint open annuli.
Step 2: Let \( v \) be a smooth vector field on \( X \) and \( \psi_t \) the flow. This determines a family \( \mathcal{A}_t \) of annuli. We would like to show that

\[
\frac{dm(\mathcal{A}_t)}{dt} \bigg|_{t=0} = 0.
\]

To do this we will show that

\[
\frac{dm(\mathcal{A}_t)}{dt} \bigg|_{t=0} = \int_X \Phi v_z = 0,
\]

where \( \Phi \) is a quadratic differential (not necessarily holomorphic) and \( v_z \) is a \((-1,1)\)-form.

Step 3: If for all smooth vector fields \( v \), \( \int_X \Phi v_z = 0 \), then \( \phi \) is actually holomorphic.

The key step is to apply the following lemma for forms instead of functions.

Lemma 4.2.4. (Weyl’s Lemma) Let \( f \) be a \( C^\infty \)-valued function on \( \Omega \subset \mathbb{C} \) (need some regularity assumptions on \( f \)). If for all \( \phi \in C^\infty_c(\Omega) \), then

\[
\int_\Omega f \phi = 0 \iff \int_\Omega f \phi = 0.
\]

Proof. (of Step 1) Let \( \mathcal{A}_n \) be a sequence of families that limits to the \( \inf \{ m(\mathcal{A}_j) \} \). The \( \inf \{ m(\mathcal{A}_j) \} \) is not \( \infty \) since \( m(A_i) \not\to 0 \), thus \( m(A_{i,j}) \not\to 0 \) as \( j \to \infty \). Let \( g_{i,j} \) be the conformal map from a standard annulus to \( A_{i,j} \). The maps \( g_{i,j} \to g_i \) (after maybe passing to a subsequence) determine a family \( \mathcal{A} \) that realizes the \( \inf \).

Proof. (of Step 2) Let \( \mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k) \) and define a quadratic differential \( \Phi \) on \( X \) be letting \( \Phi \big|_X \) be the quadratic differential given by a Euclidean structure on \( A_i \) with height \( h_i \) and \( \Phi \big|_{X - \{A_1, \ldots, A_k\}} = 0 \). Now, \( \Phi \) is a \((2,0)\) differential. If \( \mu \) is a \((-1,1)\)-differential, then \( \Phi \mu \) is a \((1,1)\)-differential and \( \int_X \Phi \mu \) is defined. Let \( v \) be a smooth \((-1,1)\)-form (AKA a vector field). Then \( v = v(z) \frac{\partial}{\partial z} \) in local coordinates. Note \( v_z \) is a \((-1,1)\)-form. Let \( \psi_t \) be the flow of \( v \). We want to calculate

\[
\frac{dm(\mathcal{A}_t)}{dt} \bigg|_{t=0} = \int_X \Phi v_z.
\]

Let \( \mathcal{A}_t \) be the collection of annuli given by the images of the standard annulus through \( \psi_t \circ g_i \) for every \( i \) (see Figure 4.2).

The modulus of the annulus \( \mathcal{A}_t \) is

\[
m(\mathcal{A}_t) = \frac{h}{c(t)} \implies c(t) = \frac{h}{m(\mathcal{A}_t)}.
\]

Then \( g_{t^{-1}} \circ \psi_t \circ g_0 = \tilde{\psi}_t \). The \( t = 0 \) derivative of \( \tilde{\psi}_t \) is a vector field \( \tilde{\nu} \) and \( \tilde{v}_z = v_z \) since all maps were conformal. In particular,

\[
\int_A \Phi v_z = \int_0^h \int_0^{c(0)} 1 \cdot \tilde{v}_z.
\]
4.2. BRANCHED DOUBLE COVERS

Figure 4.2: The annulus $A_t$.

Figure 4.3: The map $g_t^{-1} \circ \psi_t \circ g_0 = \tilde{\psi}_t$. 
Now, \( \tilde{\psi}_t(x + iy + c(0)) = \tilde{\psi}(x + iy) + c(t) \), \( Im(\tilde{\psi}_t(x)) = 0 \), \( Im(\tilde{\psi}_t(x + ih)) = h \), and \( \tilde{v}_z = \frac{1}{2}(\tilde{v}_x + i\tilde{v}_y) \). Thus,
\[
\int_0^h \int_0^{c(0)} \tilde{v}_x \, dx \, dy = \int_0^h (\tilde{v}(c(0) + iy) - \tilde{v}(iy)) \, dy = \int_0^h c(t) \, dy = -hc'(0),
\]
since
\[
\tilde{v} = \frac{d\tilde{\psi}}{dt}|_{t=0} = \frac{d}{dt}(\tilde{\psi}_t(x + c(0) + iy) - \tilde{\psi}(x + iy) + c(t)) = \tilde{v}_i(x + c(0) + iy) - \tilde{v}(x + iy) - c'(0).
\]
Moreover,
\[
\int_0^{c(0)} \int_0^h \tilde{v}_y \, dy \, dx = \int_0^{c(0)} \tilde{v}(x + hi) - \tilde{v}(x) \, dx = 0
\]
and you get
\[
\int \int 1 \cdot \tilde{v}_z = -\frac{1}{2}h \cdot c'(0) = \frac{h^2m'(A)}{2m(A)^2}.
\]
Now,
\[
m(A_t) = \frac{h}{c(t)} \Rightarrow c(t) = \frac{h}{m(A_t)} \Rightarrow c'(0) = -\frac{m'(A_0)}{m(A_0)^2} \cdot h.
\]
Thus,
\[
\frac{d}{dt} \left( \sum \frac{h_i^2}{m(A_{i,t})} \right) = 2 \int_X \Phi v_z.
\]

Now, our goal is to show that \( T(S) \) is not non-positively curved.

**Theorem 4.2.5** (Masur). Let \( X \in T(S) \). Then there exist geodesic rays \( r_t, q_t \) so that \( r_0 = q_0 \) and \( d_T(r_t, q_t) \leq B \) for all \( t \in \mathbb{R} \).

**Corollary 4.2.6.** \( T(S) \) is not non-positively curved.

**Proof.** Pick disjoint simple closed curves \( \gamma_1, \gamma_2 \) and choose Strable differentials \( \Phi_1, \Phi_2 \) so that their heights are \( 1, 1 \) in \( \Phi_1 \) and \( 1, 2 \) in \( \Phi_2 \) (see Figure 4.4).

Let \( N^1_\epsilon \) be an \( \epsilon \)-neighborhood of \( \gamma_1, \gamma_2 \) in the singular Euclidean metric defined by \( \Phi_1 \), and let \( C^1_\epsilon = \Phi_1 \setminus N^1_\epsilon \). Similarly, let \( N^2_\epsilon \) be an \( \epsilon \)-neighborhood of \( \gamma_1, \gamma_2 \) in the singular Euclidean metric defined by \( \Phi_2 \), and let \( C^2_\epsilon = \Phi_2 \setminus N^2_\epsilon \). Then \( C^1_\epsilon \) is a disjoint union of two annuli each of which has height \( 1 - 2\epsilon \) and \( C^2_\epsilon \) is a union of two disjoint annuli with heights \( 1 - 2\epsilon \) and \( 2 - 2\epsilon \). Define a map \( f_0 : \Phi_1 \to \Phi_2 \) homotopic to the identity so that \( f_0 \) takes \( N^1_\epsilon \) to \( N^2_\epsilon \), and \( f_0 \) is an affine map when restricted to each annulus of \( C^1_\epsilon \) taking them to the annuli of \( C^2_\epsilon \). Now, let \( r_t \) be the geodesic in \( T(S) \) defined by stretching \( \Phi_1 \) in the \( y \)-direction by \( t \). Let \( g_t \) be the Teichmüller geodesic defined by stretching \( \Phi_2 \) in the \( y \)-direction by \( t \). Let \( f_t : r_t \to g_t \) be equal to \( f_0 \) when restricted to \( N^1_\epsilon \) and the affine map on the complementary annuli. Now, notice that the quasi-conformal constant of \( f_t \) is bounded above by the quasi-conformal constant of \( f_0 \) (on \( N^1_\epsilon \) they are the same and on the rest \( f_t \) stretches less than \( f_0 \), therefore \( d(r_t, g_t) \leq \log(K(f_t)) \leq \log(K(f_0)) \).
Figure 4.4: Masur’s example of rays in $\mathcal{T}(S)$ emanating from a single source and staying a bounded distance from each other.


Chapter 5

The Measurable Riemann Mapping Theorem

Definition 5.0.7. Let $B(\hat{C})$ denote the set of complex valued measurable functions $\mu$ on $\hat{C}$ with $\|\mu\|_\infty < \infty$ and denote $B_1(\hat{C}) = \{ \mu \in B(\hat{C}) \mid \|\mu\|_\infty < 1 \}$. Let $B(\mathbb{H}^2)$ denote the set of complex valued measurable functions $\mu$ on $\mathbb{H}^2$ with $\|\mu\|_\infty < \infty$ and denote $B_1(\mathbb{H}^2) = \{ \mu \in B(\mathbb{H}^2) \mid \|\mu\|_\infty < 1 \}$.

We have seen that if $f : \mathbb{C} \to \mathbb{C}$ is a $K$ quasi-conformal map then $\frac{f_z}{f_z} \in B_1(\mathbb{C})$ and $\left\| \frac{f_z}{f_z} \right\| < \frac{K-1}{K+1}$. Miraculously, the converse is also true.

The Measurable Riemann Mapping Theorem (Morrey, Ahlfors-Bers, Glutysk). Given $\mu \in B_1(\hat{C})$, there exists a unique quasi-conformal map $f : \hat{C} \to \hat{C}$ such that

1. $\frac{f_z}{f_z} = \mu$
2. $f(0) = 0, f(1) = 1$ and $f(\infty) = \infty$.

We call $f$ the standard solution for $\mu$ given by the Measurable Reimann Mapping Theorem.

Corollary 5.0.8. Given a $\mu \in B_1(\mathbb{H}^2)$ where $\mathbb{H}^2$ denotes the hyperbolic plane in the half plane model, there exists unique quasi-conformal map $f : \mathbb{H}^2 \to \mathbb{H}^2$ such that

1. $\frac{f_z}{f_z} = \mu$
2. $f$ extends continuously to $\partial \mathbb{H}^2$ and $f(0) = 0, f(1) = 1$ and $f(\infty) = \infty$.

Proof. We note that any quasi-conformal map $f : \mathbb{H}^2 \to \mathbb{H}^2$ extends continuously to the boundary. We wish to invoke the Measurable Reimann Mapping Theorem. Define

$$\nu(z) = \begin{cases} 
\mu(z), & \text{if } \text{Im}(z) > 0 \\
\frac{\mu(\bar{z})}{\mu(z)}, & \text{if } \text{Im}(z) < 0
\end{cases}$$
Since \( \|\nu\| = |\mu| < 1 \), we get a standard map \( f : \mathbb{C} \to \mathbb{C} \) such that \( \nu = \frac{f}{g} \). We claim that \( f : \mathbb{H}^2 \to \mathbb{H}^2 \). Define \( g(z) = \overline{f(\bar{z})} \). It is easy to verify that \( \frac{g}{g_1}(z) = \frac{f}{f_1}(\bar{z}) = \frac{\nu(z)}{\nu(z)} \). Moreover, \( g(0) = \overline{f(0)} = 0 \), \( g(1) = \overline{f(1)} = 1 \) and \( g(\infty) = \overline{f(\infty)} = \infty \). Therefore, \( g \) is another normalized solution for \( \nu \). By uniqueness, it follows that \( f(z) = g(z) = \overline{f(\bar{z})} \). Thus for all \( r \in \mathbb{R} \), \( f(r) = \overline{f(r)} \) hence \( f \) takes the reals to the reals, and since it is quasi-conformal it is orientation preserving, so \( f \) takes the upper half plane to itself.

Let \( f : X \to Y \) be a quasi-conformal map between Riemann surfaces. Let \((U, \phi)\) be a chart in \( X \), and \((V, \psi)\) be a chart in \( Y \). Let \( f_{U,V} \) be the map \( f \) in local coordinates. We wish to understand how \( \mu_{U,V} = \frac{(f_{U,V})_*}{(f_{U,V})_*} \) changes as we vary the charts in \( U \) and \( V \).

**Observation 5.0.9.** 1. Let \((V', \psi')\) be another chart with \( V \cap V' \neq \emptyset \). One can verify that \( \mu_{U,V'} = \mu_{U,V} \). Morally, this is because the quasi-conformal dilation and directions of maximal stretch don’t change and these are exactly the absolute value and argument of \( \mu_{U,V} \). We therefore denote \( \mu_{U,V} \) simply by \( \mu_U \).

2. Let \((U', \phi')\) be another chart on \( X \) with \( U \cap U' \neq \emptyset \). Let \( \chi = \phi' \circ \phi^{-1} : \phi(U) \to \phi'(U') \). Since \( f_U = f_{U'} \circ \chi \) we get \( \mu_U(z) = \mu_{U'}(\chi(z)) \frac{\overline{\chi(z)}}{\overline{\chi(z)}} \). Therefore, \( \mu(z) \) is a \((-1,1)\) form! In addition, \( |\mu| = |\mu_U| \) is independent of the chart. Thus \( |\mu| \) is a function on \( X \) and we denote its essential sup by \( \|\mu\|_\infty \).

**Definition 5.0.10.** \( B(X) \) is the set of \((-1,1)\) forms on \( X \) with \( \|\mu\| < \infty \) and \( B_1(X) \) is the set of \((-1,1)\) forms on \( X \) with \( \|\mu\| < 1 \).

### 5.1 Bers’ Simultaneous Uniformization

**Theorem 5.1.1.** Let \( X \) be a Riemann surface and \( \mu \in B_1(X) \). Then there exists a Riemann surface \( Y \) and a quasi-conformal map \( f : X \to Y \) such that \( \mu_f = \mu \).

**Proof.** \( \bar{X} \cong \mathbb{H}^2 \) let \( \tau : \pi_1(S) \to \text{PSL}_2(\mathbb{R}) \) denote the deck transformation action. Lift \( \mu \) to \( \tilde{\mu} \in B_1(\mathbb{H}^2) \) (we can do this since \( X \) has a single chart so \( \mu \) can be thought of as a function). Let \( \tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2 \) be the normalized solution of \( \tilde{\mu} \). By the change of coordinates formula, for each \( \gamma \) we have:

\[
\tilde{\mu}(\tau(\gamma)(z)) \frac{\overline{\tau(\gamma)'(z)}}{\overline{\tau(\gamma)'(z)}} = \tilde{\mu}(z)
\]

Let \( \tilde{g}_\gamma(z) = f \circ \tau(\gamma)(z) \). By a direct calculation we get \( \mu_{\tilde{g}_\gamma} = \tilde{\mu} \). So \( \tilde{g}_\gamma \) is another solution for \( \tilde{\mu} \). For each \( \gamma \) let \( \rho(\gamma^{-1}) \) be the Mobius transformation taking \( \tilde{g}(0), \tilde{g}(1), \tilde{g}(\infty) \) to \( 0, 1, \infty \). Then \( \mu_{\rho(\gamma^{-1})} = \tilde{\mu} \) since post composing with a conformal map doesn’t change the Beltrami differential. \( \rho(\gamma^{-1}) \circ \tilde{g} = \rho(\gamma^{-1}) \circ \tilde{f} \circ \tau(\gamma) \) is a normalized solution for \( \tilde{\mu} \). By uniqueness we get \( \tilde{f} = \rho(\gamma^{-1}) \circ \tilde{f} \circ \tau(\gamma) \) or \( \rho(\gamma^{-1}) = \tilde{f} \circ \tau(\gamma) \circ \tilde{f}^{-1} \). We get that \( \rho : \pi_1(S) \to \text{PSL}_2(\mathbb{R}) \) is a representation and that it is topologically conjugate to \( \tau \). Let \( Y = \mathbb{H}^2 / \rho \) then \( \tilde{f} \) descends to a map \( f : X \to Y \) with \( \mu_f = \mu \), so \( f \) is a quasi-conformal map. 

\( \square \)
Let $\mu \in B_1(\hat{\mathbb{C}})$ define a path $\Delta \to B_1(\hat{\mathbb{C}})$ by mapping $\lambda \mapsto \lambda \mu$. Let $f_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the normalized solution. Now, fix $z$ and let $\phi_z(\lambda) = f_\lambda(z)$.

**Theorem 5.1.2.** $\phi_z(\lambda)$ is a holomorphic function and

$$\phi_z'(0) = -\frac{1}{\pi} \int_{\mathbb{C}} \mu(w) \frac{z(z - 1)}{w(w - 1)(w - z)} \, dx \, dy$$

where $w$ is the variable of integration and $w = x + iy$.

Let $X = \mathbb{H}^2/\Gamma$ where $\Gamma \subset PSL_2 \mathbb{R} \subset PSL_2(\mathbb{C})$. To recall the proof we took $\mu$ and lifted it to $\tilde{\mu}$ on $\tilde{X} = \mathbb{H}^2$ together with an equivariance property

$$\tilde{\mu}(\gamma(z)) \frac{\gamma(z)}{\gamma'(z)} = \tilde{\mu}(z) \quad (*)$$

that holds for all $z \in \mathbb{H}^2$. We also saw that for $\mathbb{H}^2 \subset \hat{\mathbb{C}}$ we can extend $\mu$ such that $(*)$ holds for all $z \in \hat{\mathbb{C}}$. Some possible extensions are

1. $\nu(z) = \begin{cases} \mu(z), & \text{if } \text{Im}(z) > 0 \\ \overline{\mu(z)}, & \text{if } \text{Im}(z) < 0 \end{cases}$
2. $\nu(z) = \begin{cases} \mu(z), & \text{if } \text{Im}(z) > 0 \\ 0, & \text{if } \text{Im}(z) < 0 \end{cases}$

We also defined $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$ and there exists $\rho : \Gamma \to PSL_2 \mathbb{R}$ such that

$$\tilde{f} \circ \gamma = \rho(\gamma) \circ \tilde{f} \quad \forall \gamma \in \Gamma$$

where $\rho$ is a representation of $\Gamma = \pi_1(S)$.

Let $\tilde{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the normalized solution for $\nu$, then there exists $\rho : \Gamma \to PSL_2 \mathbb{C}$ such that $\tilde{f} \circ \gamma = \rho(\gamma) \circ \tilde{f}$. The proof of this follows exactly the proof given before.

Let $U = \{\text{upper half plane}\}$ and $L = \{\text{lower half plane}\}$, then if $U/\Gamma = X$ and $L/\Gamma = \tilde{X}$ the actions are conjugate by the map $z \mapsto \overline{z}$. $\tilde{X}$ is really $X$ with the opposite orientation. In fact $\tilde{f}$ descends to a homeomorphism $f^+ : X \to \tilde{f}(U)/\rho(\Gamma)$ and $f^- : \tilde{X} \to \tilde{f}(L)/\rho(\Gamma)$. $\nu$ descends to Beltrami differentials $\nu^+, \nu^-$ on $X, \tilde{X}$ such that $\mu_{f^+} = \nu^+$ and $\mu_{f^-} = \nu^-$.  

**Definition 5.1.3.** A discrete subgroup of $PSL_2(\mathbb{R})$ is a *Fuchsian group* and a discrete subgroup of $PSL_2(\mathbb{C})$ is a *Kleinian group*.

Let $\nu^+ \in B_1(X)$ and $\nu^- \in B_1(\tilde{X})$ and lift them to $\nu \in B(\hat{\mathbb{C}})$. Consider the normalized solution $\tilde{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $Y^+ = \tilde{f}(U)/\rho(\Gamma)$ and $Y^- = \tilde{f}(L)/\rho(\Gamma)$. Then we get $f^+ : X \to Y^+$ and $f^- : \tilde{X} \to Y^-$ with $\mu_{f^+} = \nu^+$ and $\mu_{f^-} = \nu^-$. 

**Definition 5.1.4.** A Kleinian group $\Gamma$ is *quasi-Fuchsian* if there exists a Fuchsian group $\Gamma'$ and a quasi-conformal map $\tilde{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\Gamma = \tilde{f} \circ \Gamma_0 \circ \tilde{f}^{-1}$.

**Theorem 5.1.5.** (Bers’ Simultaneous Uniformization Theorem) For any $X \in \mathcal{T}(S)$ and $Y \in \mathcal{T}(\tilde{S})$ there is a unique quasi-fuchsian group uniformizing $X$ and $Y$.

This theorem implies that $QF(S) = \mathcal{T}(S) \times \mathcal{T}(\tilde{S})$. Philosophically, this will give us a complex structure on $\mathcal{T}(S)$. 
5.1.1 Conformal Maps

Let $\Omega \subset \hat{\mathbb{C}}$ and $f : \Omega \to \hat{\mathbb{C}}$ be a holomorphic map with $f'(z) \neq 0$ for all $z \in \Omega$ (this implies it is a local homeomorphism). We want to estimate how close $f$ is to a Möbius transformation (think that this is similar to curvature). The key is to approximate $f$ at a point by Möbius transformations and then to see how they differ.

**Fact 5.1.6.** Given $w \in \Omega$ there exists a unique $\phi \in \text{PSL}_2(\mathbb{C})$ such that $f(w) = \phi(w), f'(w) = \phi'(w),$ and $f''(w) = \phi''(w).$ The fancy way to say that the derivatives match up to the second derivative is that the 2-jet of $f$ and $\phi$ agree at $w$.

Define $M_f : \Omega \to \text{PSL}_2(\mathbb{C})$ where $M_f(w)$ is the Möbius transformation that best approximates $f$ at $w$. If $f$ is Möbius, then $M_f$ is a constant function. $M_f$ will turn out to be holomorphic. Now, let’s think about the Lie Algebra of $\text{PSL}_2(\mathbb{C}).$ Suppose $\phi_t \in \text{PSL}_2(\mathbb{C})$ is a path with $\phi_0 = \text{id}.$ Then,

$$\phi_t(z) = \frac{a(t)z + b(t)}{c(t)z + d(t)} \quad \text{and} \quad \phi'(z) = (\dot{a}z + \dot{b}) - z(\dot{c}z + \dot{d}) \text{ at } t = 0.$$ 

This last formula gives a vector field. $\phi_t^{-1} \circ \phi_t$ is a path at the identity and it will give the best approximation at time $t$ since translating back to $\text{id}$. Let $v(z) \frac{\partial}{\partial z}$ be a holomorphic vector field on $\Omega$.

Now, $M_v(w) = (v(w) + v'(w)(z - w) + \frac{v''(w)(z - w)^2}{2}) \frac{\partial}{\partial z}$ and $M_t : \Omega \to \text{sl}_2(\mathbb{C})$ ($\text{sl}_2(\mathbb{C})$ is the Lie Algebra of $\text{SL}_2(\mathbb{C})$ and so of $\text{PSL}_2(\mathbb{C})$). Now, by differentiating with respect to $w,$ we get

$$M_v'(w) = \frac{v''(w)(z - w)^2}{2} + \frac{v''(w)(z - w)^2}{2} - v''(w)(z - w).$$ 

This will have a fixed point at $w$ and $\frac{v''(w)(z - w)^2}{2}$ is holomorphic. Now, let $f : \Omega \to \mathbb{C}$ be holomorphic. If we think $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2,$ then the condition $D_x f = (a + ib)D_x \frac{\partial}{\partial z} f$ is the condition for a function to be holomorphic with $v = (a, b).$ Hence, the directional derivative is completely determined by the derivative in the $x$-direction. Let $f : \Omega \to \mathbb{C}.$ Then

$$M_f(w) = S_f(w)(z - w)^2 \frac{\partial}{\partial z}$$ 

and $S_f(w)$ is called the *Schwarzian derivative*. If you want you could calculate $S_f(w)$ explicitly. If $\phi \in \text{PSL}_2(\mathbb{C}),$ then $M_{\phi \circ f} = \phi \circ M_f$ and $M_{f \circ \phi} = M_f \circ \phi.$ The point is that $M_f' = M_{\phi \circ f}'$, but this is not true for $M_{f \circ \phi}.$ Indeed, if $\phi_0 \alpha_t \in \text{PSL}_2 \mathbb{C},$ then

$$\phi^{-1} \circ \alpha_0^{-1} \circ \alpha_t \circ \phi = (M_f \circ \phi)_0^{-1} \circ (M_f \circ \phi)_t$$ 

and this is some kind of adjoint action of the Lie group on the Lie Algebra.

Now, shift our attention back to Beltrami Differentials. Let $\mu \in B_1(\bar{X})$ and lift it to $\hat{\mathbb{C}}.$ Extend $\mu$ to zero on $\mathcal{U}$ to get $\tilde{\mu}.$ Let $\tilde{f}$ be the normalized solution for $\tilde{\mu}.$ $\tilde{f}$ is conformal on $\mathcal{U}.$ For $\gamma \in \Gamma,$

$$\tilde{f} \circ \gamma = \rho(\gamma) \circ \tilde{f} \iff \tilde{f} = \rho(\gamma) \circ \tilde{f} \circ \gamma^{-1}.$$
Let $S\tilde{f}$ be the Schwarzian derivative of $\tilde{f}$. Then
\[
S\tilde{f}(\gamma(z)) = S(\rho(\gamma) \circ \tilde{f} \circ \gamma^{-1})(\gamma(z)).
\]
Now, by our earlier computation
\[
S\tilde{f}(z)\gamma'(z)^2 = S\tilde{f}(\gamma(z))
\]
and $S\tilde{f}$ descends to a holomorphic quadratic differential on $X$. This gives a map $\mathcal{T}(S) \to Q(X)$, where the space of quadratic differentials is a complex vector space, and puts a complex structure on $\mathcal{T}(S)$ (since it is injective of the right dimension).

### 5.1.2 Bers’ Embedding

We have a map $\mathcal{T}(S) \to Q(X)$ called Bers’ embedding and $\dim\mathcal{T}(S) = \dim Q(X)$. Since $Q(X)$ is a complex vector space, the Bers’ embedding gives $\mathcal{T}(S)$ and complex structure (and so a differentiable structure). Is the structure independent on $X$? This question is the same to ask if given the embedding $\mathcal{T}(S) \to Q(X)$ and $\mathcal{T}(S) \to Q(Y)$ there is a holomorphic map $Q(X) \to Q(Y)$. We will see that the answer to this question is yes.

Let $\mu \in B_1(\overline{X})$. We lift $\mu$ to $\tilde{\mu}$ on the lower plane and we extend to 0 on the upper half plane. Let $\tilde{f}$ be the normalized solution to the Bers’ equation and consider the Schwarzian derivative $S\tilde{f} \in Q(X)$ in $X$. We defined a map $B_X : B_1(\overline{X}) \to Q(X)$. Notice that $B_1(\overline{X})$ is infinite dimensional, while $Q(X)$ is finite dimensional.

**Theorem 5.1.7.** Let $\mu_0, \mu_1 \in B_1(\overline{X})$ with solutions $f_0 : \overline{X} \to Y$ and $f_1 : \overline{X} \to Y$ suth that $f_0 \sim f_1$. Then $B_X(\mu_0) = B_X(\mu_1)$.

**Proof.** We lift the maps $f_0$ and $f_1$ to $\tilde{f}_0 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and $\tilde{f}_1 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ respectively. Since $f_0 \sim f_1$, there exists a map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

\[
\begin{array}{c}
\hat{\mathbb{C}} \\
\downarrow f_0 \\
\hat{\mathbb{C}} \\
\downarrow g \\
\hat{\mathbb{C}} \\
\end{array}
\quad \begin{array}{c}
\hat{\mathbb{C}} \\
\downarrow \tilde{f}_1 \\
\hat{\mathbb{C}} \\
\end{array}
\]

and $g$ is conformal on $f_0$ (upper half plane) and $f_0$ (lower half plane).

It is not difficult to prove that $g$ is quasi-conformal.

Since $g$ is quasi-conformal and and $g$ is conformal on $f_0$ (upper half plane) and $f_0$ (upper half plane), $g$ is conformal. If $g$ is a Möbius map, then $\tilde{f}_0 = \tilde{f}_1$ on the upper half plane.

We have $B_X : B_1(\overline{X}) \to Q(X)$ and $T_0(B_1(\overline{X})) = B(\overline{X})$. If $\mu \in B(\overline{X})$, then, for small $t \in \mathbb{R}$, $t\mu \in B_1(\overline{X})$ and $B_X(t\mu)$ is a path in $Q(X)$. Let $\dot{B}_X(\mu)$ be the tangent vector to the path $B_X(t\mu)$ at $t = 0$, hence $\dot{B}_X(\mu) \in T_0(Q(X)) = Q(X)$. We lift $\mu$ to $\dot{\mu}$, let $\tilde{f}_t$ be the normalized solution at $t\mu$. For a fixed $z$, $\tilde{f}_t(z)$ is a path in $\hat{\mathbb{C}}$. Let $\hat{f}_\mu(z)$
be the tangent vector at \( t = 0 \).

By the Measurable Riemann Mapping Theorem,

\[
\dot{f}_\mu(z) = -\frac{1}{\pi} \int \bar{\mu}(w) \frac{z(z-1)}{w(w-1)(w-z)} dx
dy,
\]

where \( w = x + iy \).

\( \dot{f}_\mu(z) \frac{\partial}{\partial z} \) is a vector field which conformal on the upper plane. By the equality 5.1, we have

\[
\dot{B}_X(\mu) = \dot{f}_\mu(3) dz^2 = \left( -\frac{6}{\pi} \int \bar{\mu}(w) \frac{1}{(w-z)^4} dx
dy \right) dz^2.
\]

What is \( \ker(B_X) \)?

Notice that if \( \mu \) is a \((-1,1)\)-differential and \( \Phi \) is a \((2,0)\)-differential, then \( \mu \Phi \) is a \((1,1)\)-differential and we can integrate. Hence, it is well defined the pairing

\[
(\mu, \Phi) \mapsto \int_X \mu \Phi
\]

Let

\[
N(X) := \{ \mu \in B(X) \mid \int_X \mu \Phi = 0, \forall \Phi \in Q(X) \} \subset B(X).
\]

The pairing we have defined induces a pairing on the quotient:

\[
B(X)/N(X) \times Q(X) \to \mathbb{C}.
\]

Since we killed all the elements in \( B(X) \) that go to 0,

\[
dim(B(X)/N(X)) \leq dim(Q(X)).
\]

Moreover, we have the following

**Lemma 5.1.8.**

\[
dim(B(X)/N(X)) = dim(Q(X)).
\]

**Proof.** If \( \Phi \in Q(X) \), the \( \frac{\Phi}{|\Phi|} \in B(X) \).

Since if \( \Phi \neq 0 \), then

\[
\int_X \frac{\overline{\Phi}}{|\Phi|} \Phi = \int_X |\Phi| > 0
\]

and therefore, \( dim(B(X)/N(X)) \geq dim(Q(X)) \).

We want to show that \( \dot{B}_X : B(\mathbb{H})/N(\mathbb{H}) \to Q(X) \) is an isomorphism, so that we can identify \( B(\mathbb{H})/N(\mathbb{H}) \) with the tangent space \( T_X(T(S)) \).

In order to do that, we need to write \( N(X) \) in another way.

Let \( \mu \in B(X) \) and lift \( \mu \) to \( \tilde{\mu} \in B(\mathbb{H}^2) \). By the infinitesimal version of the Riemann Mapping Theorem, there exists a vector field \( \tilde{v}(z) \frac{\partial}{\partial z} \) in \( \mathbb{H}^2 \) such that \( \tilde{v}_z = \tilde{\mu} \). We have \( X = \mathbb{H}^2/\Gamma \) and

\[
\tilde{\mu}(\gamma(z)) \frac{\gamma'(z)}{\gamma'(z)} = \tilde{\mu}(z).
\]
Hence, we have

\[
\frac{\partial}{\partial \bar{z}}(\gamma'(z)\bar{v}(z) - \bar{v}(\gamma(z))) = \gamma'(z)\bar{v}_z(z) - \bar{v}_z(\gamma(z))\gamma'(z) = \\
\gamma'(z)\bar{\mu}(z) - \bar{\mu}(\gamma(z))\gamma'(z) = 0
\]

and so the vector field \(\gamma'(z)\bar{v}(z) - \bar{v}(\gamma(z))\) is holomorphic.

Moreover, notice that \(\gamma'(z)\bar{v}(z) - \bar{v}(\gamma(z)) \in \text{sl}_2(\mathbb{R})\).

Since the vector field is holomorphic, it descends to a vector field on \(X\). When we flow the vector field in \(X\) we have a map homotopic to the identity, so we have a trivial deformation in the Teichmüller space.

Now, we consider \(X\) to be the annulus in Figure 5.1.

**Figure 5.1: The annulus \(X\).**

Consider \(<\gamma> = \Gamma = \mathbb{Z}\). where \(\gamma(z) = \lambda z, \lambda \in \mathbb{R}\). Then

\[
\bar{\mu}(\gamma^n(z)) = \bar{\mu}(\lambda^n z) = \bar{\mu}(z).
\]

**Fact 5.1.9.** If \(\Gamma \subset PSL_2(\mathbb{C})\), \(\mu \in B_1(\mathbb{C})\) such that

\[
\mu(\gamma(z))\frac{\gamma'(z)}{\gamma'(z)} = \mu(z),
\]

and \(f\) is the normalized solution, then \(f \circ \Gamma \circ f^{-1} \subset PSL_2(\mathbb{C})\).

Let \(\gamma_t(z) = \lambda tz\) and \(\tilde{f}_t\) be the normalized solution in \(\mathbb{H}^2\). By the Fact 5.1.9,

\[
\tilde{f}_t \circ \gamma_t \tilde{f}_t^{-1} = \gamma_t \in PSL_2(\mathbb{R}).
\]

Let \(\tilde{v}(z) = \tilde{f}_t\mu(z)\). Consider the flow given by \(\gamma_t\), take the derivative and denote it by \(\hat{\gamma}(z) \in \text{sl}_2(\mathbb{R})\). If we differentiate \(\tilde{f}_t \circ \gamma_t \tilde{f}_t^{-1} = \gamma_t\), we get

\[
\gamma'(z)\tilde{v}(z) - \tilde{v}(\gamma(z)) = \hat{\gamma}(z).
\]

If it is equivariant, i.e. equal 0, then we have a trivial deformation in the Teichmüller space. If it is not equivariant, i.e. not equal 0, then we have a non-trivial deformation in the Teichmüller space.

We have the following
Theorem 5.1.10. \( \mu \in N(X) \) if and only if there exists a vector field \( v \) on \( X \) such that \( v_z = \mu \).

Proof. \( \Leftarrow \): Let \( \Phi \in Q(X) \). By the assumption and the integration by parts, we have

\[
\int_X \mu \Phi = \int_X v_z \Phi = -\int_X v \Phi_z = 0,
\]

since \( \Phi \) is holomorphic (i.e. \( \Phi_z = 0 \)).

\( \Rightarrow \): Let \( \mu \in B(X) \) a Beltrami differential which does not arise as the \( \bar{z} \) derivative of any vector field of \( X \). We wish to prove that \( \mu \notin N(X) \). Lift \( \mu \) to \( \tilde{\mu} \in B(\mathbb{H}^2) \). Let \( \tilde{v} \) be the unique (up to normalization) vector field such that \( \tilde{v}_z = \tilde{\mu} \). Let \( X = \mathbb{H}^2/\Gamma \) since we’re assuming that \( \tilde{v} \) does not descend to a vector field on \( X \), there exists an element \( \gamma \in \Gamma \) such that

\[
\gamma'(z) \tilde{v} - \tilde{v}(\gamma(z)) \neq 0
\]

Recall the above expression is an element of \( \text{sl}_2(\mathbb{R}) \). We can normalize the group \( \Gamma \) so that \( \gamma(z) = \lambda z \) for some \( \lambda > 0 \).

Extend \( \tilde{\mu} \) to \( \tilde{C} \) by letting \( \tilde{\mu} = 0 \) on the lower half plane. Note that for all \( \alpha \in \Gamma \) we still have:

\[
\tilde{\mu}(\alpha(z)) \frac{\alpha'(z)}{\alpha'(z)} = \tilde{\mu}(z)
\]

Indeed it is true for all \( \alpha \in \Gamma \) in the upper half plane since \( \tilde{\mu} \) is the lift of a Beltrami differential on \( X \). And it is true for \( z \) in the lower half plane since \( \alpha \in \text{Gamma} \) preserves the lower half plane and \( \tilde{\mu} \) is identically 0 there.

Now, \( T_\gamma = \tilde{C} \setminus \{0, \infty\} \backslash < \gamma > \) is a torus. \( \tilde{\mu} \) descends to \( \mu_\gamma \) in \( B(T_\gamma) \). On the torus there is a unique quadratic differential up to scale which is the one descending from \( \tilde{\phi}(z) = \frac{1}{z^2} \). Indeed,

\[
\tilde{\phi}(\gamma^n(z))(\gamma'(z))^2 = \tilde{\phi}(z)
\]

Let \( \phi_\gamma \) denote the quadratic differential of \( T_\gamma \) that \( \tilde{\phi} \) descends to.

Now, comes a calculation that \( \int_{T_\gamma} \mu_\gamma \phi_\gamma \neq 0 \). To make this calculation, consider \( A(1, \lambda) \) denote the annulus whose inner radius is 1 and outer radius is \( \lambda \) in \( \tilde{C} \). This is a fundamental domain for the \( < \gamma > \) action on \( \tilde{C} \setminus \{0, \infty\} \). The map \( z \to e^z \) takes the rectangle with side lengths \( 2\pi, \log \lambda \) to \( A(1, \lambda) \). Pulling back \( \frac{1}{z^2} \) we get the quadratic differential 1 on the rectangle. One can compute \( 0 \neq \int_{\text{rec}} \mu_{\text{rec}} \frac{1}{z^2} = \int_{A(1, \lambda)} \tilde{\mu} \tilde{\phi} = \int_{T_\gamma} \mu_\gamma \phi_\gamma \). The fact that \( \tilde{v} \) satisfies equation 5.2 translates to the fact that the terms don’t cancel in the first integral to show that it is non-zero.

We have that \( \int_{T_\gamma} \mu_\gamma \phi_\gamma \neq 0 \) but we must show that there is a quadratic differential \( \psi \) on \( X \) so that \( \int_X \mu \psi \neq 0 \).

Poincaré Series. Let \( \Gamma \) be any discrete subgroup of \( \text{PSL}_2(\mathbb{C}) \) (in this proof \( \Gamma = \pi(X) \) or \( < \gamma > \)). Let \( f : \mathbb{H}^2 \to \mathbb{C} \) be holomorphic. The poincare series of \( f \) is:

\[
\Theta f(z) = \sum_{\alpha \in \Gamma} f(\alpha(z))(\alpha'(z))^2
\]
This series might not converge, however there are many cases that it does, for example, if \( \int_{\mathbb{H}} |f| \, dx \, dy \) then \( \Theta f(z) \) is finite. In addition, notice that this is a way of producing quadratic differentials:

\[
\Theta f(\beta(z))\beta'(z)^2 = \sum_{\alpha \in \Gamma} f(\alpha \circ \beta(z))\alpha'(\beta(z))^2/\beta'(z)^2 = \Theta f(z)
\]

Let \( F \) be a fundamental domain for \( \Gamma \). Let \( \tilde{\mu} \) be a \( \Gamma \)-equivariant Beltrami differential, and let \( \tilde{\psi} = \Theta f \). By its construction \( \tilde{\psi} \) is \( \Gamma \) equivariant and descends to a quadratic differential \( \psi \) on \( \mathbb{C} \setminus \{0, 1\} / \Gamma \) or \( \mathbb{H} / \Gamma \). Note that:

\[
\int \mu \psi = \int_F \tilde{\mu} \tilde{\psi} = \int_F \tilde{\mu} \Theta f = \int_{\mathbb{C} \setminus \{0, 1\}} \text{or} \int_{\mathbb{H}} \mu f
\]

Going back to our situation, notice that we can actually write \( 1/z^2 \) as \( \Theta f \) for an appropriately chosen \( f \). Indeed, let \( f = \frac{\lambda^1}{z(z-1)(z-\lambda)} \) then \( f(z) = -\frac{1}{z} \left( \frac{1}{z-1} \frac{1}{z-\lambda} \right) \) Hence

\[
\Theta_\gamma f(z) = \sum_{n \in \mathbb{Z}} f(\lambda^n z)\lambda^{2n} = \frac{1}{z} \sum_{n \in \mathbb{Z}} \left( \frac{1}{z-\lambda^{-n}} - \frac{1}{z-\lambda^{1-n}} \right) = -\frac{1}{z^2}
\]

Therefore,

\[
0 \neq \int_{\mathcal{T}_\gamma} \mu_\gamma \phi_\gamma = \int_{\mathbb{C} \setminus \{0, 1\}} \tilde{\mu} f(z)
\]

But since \( \tilde{\mu} = 0 \) on the lower half plane

\[
\int_{\mathbb{C} \setminus \{0, 1\}} \tilde{\mu} f(z) = \int_{\mathbb{H}} \tilde{\mu} f = \int_X \mu \Theta \Gamma f \tag{5.3}
\]

Thus there exists a quadratic differential on \( X \) namely \( \Theta \Gamma f \) so that its pairing with \( \mu \) is non-zero. Thus \( \mu \not\in N(X) \). We remark that we need to worry a little about the convergence of \( \Theta \Gamma f \), but comes out from the equality in 5.3.

### 5.1.3 An addendum to the infinitesimal Measurable Riemann Mapping Theorem

We give a sketch of why the theorem is true. First let us compute the \( \bar{z} \) derivative of \( \frac{1}{z} \). One can show that for all holomorphic \( \phi \):

\[
-\frac{1}{\pi} \int \phi_\bar{z}(z) \frac{1}{z} \, dx \, dy = \phi(0)
\]

Therefore, by integration by parts, \( \frac{\partial}{\partial \bar{z}} \frac{1}{z} \) is the atomic infinite measure at 0, the Dirac delta function.

Now, given a Beltrami differential \( \mu \) we wish to find \( v \) so that \( v_\bar{z} = \mu \).

\[
v(z) = -\frac{1}{\pi} \int_C \frac{\mu(w)}{w-z} \, dx \, dy
\]
If $\mu$ is smooth with compact support we get $v_\bar{z}(z) = \mu(z)$. In PDE jargon $\frac{1}{\pi z}$ is a fundamental solution for $f_\bar{z} = \mu$.

The normalized solution of the infinitesimal measurable Riemann mapping theorem is

$$v(z) = \int_C \mu(w) \frac{z(z-1)}{w(w-1)(w - \lambda)} dx dy$$

We can simplify this expression: $\frac{z(z-1)}{w(w-1)(w - \lambda)} = \frac{1}{w-z} - \frac{z-1}{w}$. The $\bar{z}$ derivative of the last two terms is zero, so with the new formula we still get $v_\bar{z}(z) = \mu(z)$. We can also easily see that $v(0) = v(1) = 0$. It also works out that $v(\infty) = 0$.

\[ ^1 \text{Why are these the conditions?} \]
Chapter 6

The Weil-Petersson metric

We can represent infinitesimal deformations of $\mathcal{T}(S)$ as Beltrami differentials. We have proven the pseudo-theorem: $B(X)/N(X) = T_X\mathcal{T}(S)$ this is only a pseudo theorem since we haven’t defined a differential structure on $\mathcal{T}(S)$ yet. However, we may ask when does the infinitesimal deformation represented by $\mu \in B(X)$ trivial? It is iff $\mu \in N(X)$.

6.1 Norms on $Q(X)$

The cotangent bundle of $\mathcal{T}(S)$ is $Q(X) = T^*_X(\mathcal{T}(S))$. We can define several norms on it.

$\phi \in Q(X)$ is a $(2, 0)$ form. $|\phi|$ is a $(1, 1)$ form. If $\rho^2$ is the area form of a metric $\rho$, then it is a $(1, 1)$ form. Let $\rho_X$ denote the hyperbolic metric on $X$. Then $\frac{|\phi|}{\rho_X^2}$ is a function on $X$. More explicitly, let $\tilde{\phi}$ denote the lift of $\phi$ to $Q(\mathbb{H}^2)$ (the upper half plane model). Then $\frac{|\tilde{\phi}|}{\rho_{\mathbb{H}^2}} = \text{Im}(z)^2|\tilde{\phi}|$ is a function on $\mathbb{H}^2$ which we denote by $|\phi|_{\mathbb{H}^2}$.

Since $\phi$ is $\gamma$ equivariant, so is $|\phi|_{\mathbb{H}^2}$ which descends to $|\phi|_{\rho_X}$.

**Definition 6.1.1.** The $L^p$ norm of $\phi \in Q(X)$ is the $L^p$ norm of the function $|\phi|_{\rho_X}$ with respect to the hyperbolic area.

**Example 6.1.2.**

- $p = 1$: $\|\phi\|_1 = \int_X \frac{|\phi|}{\rho_X^2} \rho_X^2 = \int_X |\phi|$ which is the Euclidean area.

  This turns out to be the Teichmüller co-metric.

- $p = 2$: The norm comes from an inner product:

  $$<\phi, \psi>_2 = \int_X \frac{\phi \overline{\psi}}{\rho_X^2} \rho_X^2.$$ 

  Hence, $\|\phi\|_2 = \sqrt{\int_X \frac{|\phi|^2}{\rho_X^2}}$. This is the Weil-Peterssson co-metric.

Let $V$ be a vector space, and $V^*$ its dual. Let $(\cdot, \cdot) : V \times V^* \to \mathbb{C}$ be the pairing of $V$ with $V^*$. Given a norm $\| \cdot \|_*$ on $V^*$, one can pull it back to define a norm on $V$ as follows:

$$\|v\| = \inf_{w \in V^*} \frac{(v, w)}{\|w\|_*}.$$
In fact, all we are using in this definition is the existence of the pairing (and not the fact that it is non-degenerate). Thus we can use this definition whenever there exists such a pairing. For example, for $V = B(x)$ and $V^* = Q(x)$.

This construction yields norms on $T_x(T(S))$.

### 6.2 Incompleteness of the Weil-Petersson metric

The goal of this section is to prove the following theorem due to Wolpert and Chu.

**Theorem 6.2.1.** The Weil-Petersson metric is not a complete metric.

**Proof.** Let $X$ be a Riemann surface obtained from the square $R$ with vertices 0, 1, $i$ and $i + 1$. We assume that the two vertical sides are identified by a horizontal translation. The top an bottom are identified by some more complicated gluing pattern. Let $X_t$ be the path in Teichmüller space obtained by squeezing the rectangle to one of width $e^{-t}$. We call this rectangle $R_t$. Hence, we have a map $\tilde{f}_t : R \to R_t$, $\tilde{f}_t(x + iy) = e^{-t}x + iy$, which descends to a map $f_t : X \to X_t$. This is actually a Teichmüller geodesic but this will not be important. We will see that this path has finite Weil-Petersson length.

First, we note that a fairly straightforward calculation shows that the derivative of the path at time $t$ is the Beltrami differential $\mu_t = -1/2$, where we are writing $\mu_t$ in the coordinates given by the rectangle we are using to build the surface.

Recall that $\|\mu_t\|_{WP} = \sup_{\phi} \frac{|\int_{X_t} \mu_t \phi|}{\|\phi\|_2}$ where the sup varies over all non-zero quadratic differentials $\phi \in Q(X_t)$. Note that $\mu_t \phi$ is a form of type $(1,1)$ (essentially a 2-form) so the integral makes sense. For the $L^2$-norm $\|\phi\|_2$ we need to use the hyperbolic metric. Also note that the ratio is scale invariant.

We can assume that $\phi$ is some holomorphic function on $R_t$ and we can assume that

$$\int_{R_t} \phi = -2e^{-t},$$

where the integral is with respect to the standard Euclidean area form. Let $\phi_0 = \phi + 2$. So

$$\int_{R_t} \phi_0 = \int_{R_t} \phi + \int_{R_t} 2 = -2e^{-t} + 2e^{-t} = 0$$

and

$$\int_{X_t} \mu_t \phi = \int_{R_t} (-1/2)(\phi_0 - 2) = e^{-t}.$$

We now need to estimate $\|\phi\|_2$. Let $\rho$ be the conformal factor in $R_t$ for the hyperbolic metric on $X_t$. Note that this is a finite area metric on the rectangle and is necessarily incomplete. We can extend $\rho$ to an incomplete hyperbolic metric on the entire strip between the $\mathbb{R}$-axis and horizontal line at height $i$.

Let $\rho_S$ be the complete hyperbolic metric on the this same strip. The Schwarz Lemma
6.2. INCOMPLETENESS OF THE WEIL-PETERSSON METRIC

(of course!) implies that \( \rho_S > \rho \). We also note that by symmetry \( \rho_S \) is constant along horizontal lines. Finally, one can check to see that

\[
\int_{R_t} \phi_0 g = 0
\]

for any function \( g \) that is constant along horizontal lines. To see this we let \( C \) be a rectangular contour with vertical sides at 0 and \( e^{-t} \) and horizontal sides at heights \( ia \) and \( ib \) with \( 0 \leq a < b \leq 1 \) (see Figure 6.1). Since \( \phi \) is holomorphic, by Cauchy Theorem,

\[
\int_{C} \phi_0 dz = 0.
\]

![Figure 6.1: The contour C with vertical sides at 0 and e^{-t} and horizontal sides at heights ia and ib.](image)

The function \( \phi_0 \) will extend to a periodic function, with period \( e^{-t} \), on the entire strip between the \( \mathbb{R} \)-axis and the horizontal line at height \( i \) so the two vertical sides of the contour will cancel each other out and therefore, if \( H_a \) is the horizontal line segment of length \( e^{-t} \) starting at \( ia \), we have

\[
\int_{H_a} \phi_0 dz = \int_{H_b} \phi_0 dz = A,
\]

for all \( 0 \leq a < b \leq 1 \) (\( A \) is a constant). We then see that

\[
0 = \int_{R_t} \phi_0 = \int_{0}^{1} \int_{H_s} \phi_0 dz ds = A
\]

so we must have

\[
\int_{H_s} \phi_0 dz = 0.
\]

Notice that if

\[
\int_{H_s} \phi_0 dz = 0 \Rightarrow \int_{H_s} \bar{\phi}_0 dz = 0.
\]

If \( g \) is constant along horizontal lines then

\[
\int_{R_t} \phi_0 g = \int_{0}^{1} \int_{H_s} \phi_0 g dz ds = 0 \text{ and also } \int_{R_t} \bar{\phi}_0 g.
\]

(6.1)
We use this to bound $\|\phi\|_2$. We calculate:

$$\|\phi\|_2^2 = \int_{R_t} \frac{|\phi|^2}{\rho^2} \text{ by definition} \geq \int_{R_t} \frac{|\phi|^2}{\rho_S^2} (\text{since } \rho_S > \rho)$$

$$= \int_{R_t} \frac{|\phi_0|^2 - 2(\phi_0 + \bar{\phi}_0) + 4}{\rho_S^2} (\text{remember } \phi = \phi_0 - 2)$$

$$= \int_{R_t} \frac{|\phi_0|^2 + 4}{\rho_S^2} (\text{by equalities 6.1})$$

$$\geq \int_{R_t} \frac{4}{\rho_S^2} = e^2 e^{-t} \text{ (since } \frac{|\phi_0|^2}{\rho_S^2} \text{ is positive).}$$

So we have

$$\|\phi\|_2 \geq c e^{-t/2}$$

and

$$\frac{\left| \int_{X_t} \mu_t \phi \right|}{\|\phi\|_2} \leq e^{-t/2} = \frac{e^{-t/2}}{c}$$

for all non-zero $\phi \in Q(X_t)$. Note that we could explicitly calculate $c$ by explicitly writing down $\rho_S$. This isn’t especially difficult but also isn’t necessary so we will skip it.

Finally, we have

$$\|\mu_t\|_{WP} \leq \frac{e^{-t/2}}{c}$$

and

$$\int_0^\infty \|\mu_t\|_{WP} dt \leq \int_0^\infty e^{-t/2} \frac{c}{c} dt = \frac{2}{c} < \infty.$$ 

In particular, the path $X_t$, $0 \leq t \leq \infty$, has finite Weil-Petersson length.

### 6.3 Other properties of the Weil-Petersson metric

The Weil-Petersson metric is a Riemannian metric on $T(S)$, then there are unit speed parameterization geodesics $g : (a, b) \to T(S)$.

We have the following property.

**Theorem 6.3.1.** The map $l_\gamma \circ g : (a, b) \to (0, \infty)$ is strictly convex, where $l_\gamma$ is the length function associated to an essential simple closed curve $\gamma$.

This is a consequence of the fact that the Weil-Petersson metric is a Kähler metric (geometrically means that the metric on the tangent space approximates to order 2).

**Proof.** Let $\mu \in HB(X)$. Consider the path $\eta : (a, b) \to T(S)$, $\eta(t) = t\mu$ for $t$ small. We can compose $\eta$ with $l_\gamma$ to get a function $l_\gamma \circ \eta : (a, b) \to (0, \infty)$. Since the Weil-Petersson metric is Kähler, the sign of the second derivative of $l_\gamma \circ \eta$ is equal to the
sign of the second derivative of \( l_\gamma \circ g \) if \( g(0) = 0 \) and \( g'(0) = \mu \).

We start doing a general setup. Let \( \mu \in B(X) \). Then \( t\mu \) is a path in \( T(S) \). For fixed \( t, t\mu \in T(S) \). Consider \( l(t) := l_\gamma(t\mu) \).

We want to find a formula for \( \ddot{l}(0) \).

Let \( \Sigma := \{ z \mid 0 < \Im(z) < \pi \} \). By the Riemann Mapping Theorem, we can consider the conformal map \( \Sigma \rightarrow \mathbb{H}^2 \). The pull back of the hyperbolic metric \( \rho_{\mathbb{H}^2} \) is the hyperbolic metric \( \rho = \frac{1}{\sin(y)} \) on \( \Sigma \).

Notice that the traslation of length \( |x| \mapsto z + x, x \in \mathbb{R} \) is an isometry and

- if \( \Im(z) = \frac{\pi}{2} \), the metric is Euclidean;
- if \( |\Im(z) - \frac{\pi}{2}| > 0 \), the metric is > Euclidean.

The map \( f^t : \Sigma \rightarrow \Sigma \) has Beltrami differential \( t\mu \). We want to normalize the setting in the following way. We consider a lift \( \tilde{\gamma} \) of \( \gamma \) and we suppose that \( \tilde{\gamma} \) is as in Figure 6.2, in which there is the disc model of the hyperbolic space (that is the covering of \( X \)) and its conformal annulus. The curve \( \tilde{\gamma} \) is the red curve.

\[ F^t(0) = 0, \quad F^t(1) = 1, \quad F^t(\infty) = \infty. \]

Notice that \( F^t \) fixes \( \mathbb{R} \) and it maps the line of height \( k\pi, k \in \mathbb{Z} \) to a line parallel to \( \mathbb{R} \).

Let \( F^t \) be the normalized solution to \( t\mu \), \( F^t(0) = 0, \quad F^t(1) = 1, \quad F^t(\infty) = \infty \). Notice that \( F^t \) fixes \( \mathbb{R} \) and it maps the line of height \( k\pi, k \in \mathbb{Z} \) to a line parallel to \( \mathbb{R} \).

If we fix \( z \) and we move \( t \), then \( F^t(z) \) is holomorphic. We want to change the normalization so that we get smoothness.
Consider \( h(t) := \text{Im}(F^t(x + i\pi)) = \text{Im}(F^t(i\pi)) \).

By the Measurable Riemann Mapping Theorem, \( h(t) \) is a smooth function. By the uniqueness of the Measurable Riemann Mapping Theorem, we get the following

**Lemma 6.3.2.** \( f^t(z) = \pi \frac{F^t(z)}{h(t)} \) and \( t \mapsto f(z, t) \) is smooth for each \( z \in \mathbb{C} \).

Let \( z = x + iy \). We fix \( y \) such that \( 0 < \text{Im}(z) < \pi \) and we calculate

\[
\int_0^{l(0)} f^t_x(z) \, dx = f^t(l(0) + iy) - f^t(iy) = l(t).
\]

So we have

\[
\int_0^\pi \int_0^{l(0)} f^t(z) \, dx \, dy = \int_0^\pi l(t) = \pi l(t).
\]

Hence, if we call \( F \) the fundamental domain of the action at time \( t = 0 \) i.e., \( F = (0, l(0)) \times (0, i\pi) \), we have

\[
\int_F f^t_x(z) \, dx \, dy = \pi l(t).
\]

Now, fix \( x \). We calculate

\[
\text{Im}\left( \int_0^\pi f^t_y(z) \, dy \right) = \text{Im}(f^t(x + iy) - f^t(x)) = \text{Im}(f^t(x + iy)) = \pi.
\]

Then

\[
\int_F f^t_y(z) \, dx \, dy = \pi l(0).
\]

We know that

\[
f^t_x = \frac{1}{2}(f^t_x + if^t_y).
\]

We have

\[
\Re\left( \int_F f^t_x(x + iy) \, dx \, dy \right) = \frac{\pi}{2} (l(t) - l(0)). \tag{6.2}
\]

We want \( \dot{l}(0) \) and the idea is to use this integral representation to find it. Recall that the notations are

\[
f^t(z) = f(z, t) \quad \text{and} \quad \dot{f}^t(z) = \frac{\partial f}{\partial t}(z, t).
\]

Taking the derivative in 6.2, we have

\[
\frac{\pi}{2} \dot{l}(t) = \Re\left( \int_F \dot{f}^t_x(x + iy) \, dx \, dy \right).
\]

Since \( f^t_x = t\mu f^t_x \), \( \dot{f}^t_x = \mu f^t_x + t\mu \dot{f}^t_x \). At \( t = 0 \), \( \dot{f}^t_x = \mu \) and so at \( t = 0 \),

\[
\frac{\pi}{2} \dot{l}(0) = \Re\left( \int_F \mu \, dx \, dy \right).
\]

**Theorem 6.3.3** (Gardiner). For \( \gamma \in \pi_1(S) \) there exists \( \Phi_\gamma \in Q(X) \) such that \( \dot{l}(0) = \Re < \eta, \Phi_\gamma > \).
Since we want $\ddot{l}(0)$, we need to take another derivative.
Observe that $\dddot{\bar{z}} = 2\mu \dot{\bar{z}}^t + \mu \dddot{\bar{z}} + t\mu \dddot{\bar{z}}.$
At $t = 0$, $\dddot{\bar{z}} = 2\mu \dot{\bar{z}}^t$, then
$$\frac{\pi}{4} \ddot{l}(0) = Re\left( \int \mu \dot{z}^t \, dx \, dy \right).$$
We are going to apply this formula for $\mu \in HB(X)$.
Assume that $\mu \in HB(X)$. This means that
$$\mu(z) = \frac{1}{\rho^2} \overline{\phi(z)} = \sin^2(y)\overline{\phi(z)},$$
where $\phi$ is holomorphic with $\phi(z + l(0)) = \phi(z)$. So
$$\phi(z) = \sum_{n \in \mathbb{Z}} a_n e^{cnz} \text{ (Fourier series)}.$$
Let $\dot{f}(z) = \frac{\partial f}{\partial t}(z, 0)$. We have
1. (By the Measurable Riemann Mapping Theorem) $\dot{f}(z)$ is a continuous function on $\Sigma$;
2. (growth bound) Since $f^t(z + nl(0)) = f^t(z) + nl(t)$, $n \in \mathbb{Z}$, $|\dot{f}(x + iy)| < K|x|$;
3. $\dddot{\bar{z}} = \mu = \sin^2(y)\overline{\phi(z)}$;
4. $\dot{f}(x)$ and $\dot{f}(x + i\pi)$ are real numbers.
Let solve the equation $\dddot{\bar{z}} = \sin^2(y)\overline{\phi(z)}$. Since
$$\sin^2(y) = \frac{1}{2} - \frac{1}{4}e^{z-\bar{z}} - \frac{1}{4}e^{-z+\bar{z}},$$
we get
$$\mu(z) = \frac{1}{2} \overline{\phi(z)} - \frac{1}{4} e^z \overline{\phi(z)e^{-z}} - \frac{1}{4} e^{-z} \overline{\phi(z)e^z}.$$ 
Notice that $e^z$ and $e^{-z}$ are holomorphic functions and $\overline{\phi(z)}$, $\overline{\phi(z)e^{-z}}$ and $\overline{\phi(z)e^z}$ are anti-holomorphic functions.
Integrating $\mu(z)$ in $\bar{z}$, we have the solution
$$w(z) = \frac{1}{2} \int_{z_0}^z \phi(\tau) \, d\tau + \frac{1}{4} e^z \int_{z}^{+\infty} \phi(\tau)e^{-\tau} \, d\tau - \frac{1}{4} e^{-z} \int_{-\infty}^{z} \phi(\tau)e^{\tau} \, d\tau.$$ 
Since we can add to $w(z)$ all the holomorphic functions that we want, we change $w(z)$ to
$$\dot{f} = Re\left( \int_{z_0}^z \phi(\tau) \, d\tau \right) + \frac{1}{2} e^z Re\left( \int_{z}^{+\infty} \phi(\tau)e^{-\tau} \, d\tau \right) - \frac{1}{2} e^{-z} \left( \int_{-\infty}^{z} \phi(\tau)e^{\tau} \, d\tau \right).$$
If $w_1$ and $w_2$ are two solutions of the equation, since (3) holds for both the solutions, they are holomorphic functions. From (2) and (4), we have the uniqueness of the solution.
Now, we put $\phi(z) = \sum_{n \in \mathbb{Z}} a_n e^{cnz}$ in $\dot{f}$ and we substitute in $\mu(z) = \sin^2(y)\overline{\phi(z)}$. Doing an estimate of $\ddot{l}(0)$, one can conclude the proof of the Theorem.
Bibliography


