THE DOLD-KAN THEOREM

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ABSTRACT. These notes give a proof of the Dold-Kan theorem over an abelian category without using element arguments. We motivate the various constructions and results along the way by analogy to simplicial sets.

1. Notation

Let $\boldsymbol{\Delta}$ denote the simplex category and fix an abelian category $\mathcal{A}$. Define the category of *simplicial $\mathcal{A}$-objects* to be the functor category $s\mathcal{A} := [\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{A}]$, and $\mathrm{ch}\mathcal{A}$ to be the category of nonnegative, homologically graded chain complexes over $\mathcal{A}$.

The objects of $\boldsymbol{\Delta}$ are written $[n] = \{0, \ldots, n\}$ and the coface and codegeneracy maps will be denoted d^i and s^i. The face and degeneracy maps of a simplicial object will be denoted by d_i and s_i.

Given a map $\eta : [n] \to [m] \in \boldsymbol{\Delta}$, the corresponding map on a simplicial object will be denoted η^*. Epimorphism will be notated by $\twoheadrightarrow$, monomorphisms will be notated by $\hookrightarrow$, and isomorphisms by $\xrightarrow{\sim}$.

2. Chain Complexes from Simplicial Objects

A simplicial set is determined by its nondegenerate simplices and face maps. In a simplicial set, the degenerate simplices are the elements which appear in the image of the degeneracy maps, allowing us to isolate the nondegenerate simplices as the elements in the complement.

Given a simplicial object A in an abelian category, we can define the degenerate n-simplices DA_n analogously, and we will show that the nondegenerate simplices $NA_n = A_n/DA_n$ are indeed complimentary, i.e., that the exact sequence

$$0 \longrightarrow DA_n \longrightarrow A_n \longrightarrow NA_n \longrightarrow 0$$

splits, and moreover, that the isomorphism $A_n \cong DA_n \oplus NA_n$ is natural. As in the case of simplicial sets, keeping track of the face maps allow us to recover the original simplicial object, and doing so in this setting gives NA the structure of a chain complex.

Definition 2.1. Let $C : s\mathcal{A} \to \mathrm{ch}\mathcal{A}$ be the functor assigning each simplicial object A to its associated chain complex,

$$CA_n = A_n, \qquad \partial_n^{CA} = \sum_{i=0}^{n} (-1)^i d_i,$$

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and let $N, D : s\mathcal{A} \to ch\mathcal{A}$ be the functors taking A to the natural subcomplexes of CA

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i)$$
$$DA_n = \sum_{i=0}^{n-1} \mathrm{im}(s_i).$$

These are the *normalized (or nondegenerate)* and *degenerate* complexes respectively.

Remark 2.2. The complexes NA, DA, and CA are defined purely in terms of the face and degeneracy maps. Since a morphism of simplicial objects commutes with these maps, the inclusions $DA \hookrightarrow CA$ and $NA \hookrightarrow CA$ are natural in A, inducing a natural map $NA \oplus DA \to CA$.

Proposition 2.3. *The natural map* $NA \oplus DA \to CA$ *is an isomorphism.*

Corollary 2.4. *For each* n*, the isomorphism* $DA_n \oplus NA_n \cong A$ *is natural in* A*.*

Proof of proposition 2.3. Fix $A \in s\mathcal{A}$. We will show $NA \hookrightarrow CA \twoheadrightarrow CA/DA$ is an isomorphism.

We proceed by induction. For all $n \geqslant j \geqslant 0$, let

$$N_j A_n = \bigcap_{i=0}^{j-1} \ker d_i$$
$$D_j A_n = \sum_{i=0}^{j-1} \mathrm{im}\, s_i.$$

and define

$$\varphi_{j,n} := N_j A_n \hookrightarrow A_n \twoheadrightarrow \frac{A_n}{D_j A_n}$$

Consider the case $j = 0$. For all $n \geqslant 0$, $N_0 A_n \cong A_n$ and $D_0 A_n \cong 0$, so

$$N_0 A_n \cong \frac{A_n}{D_0 A_n}.$$

Now suppose $j \geqslant 0$, and that for all $k \leqslant j$ and $m \geqslant k$, $\varphi_{k,m}$ is an isomorphism. Fix $n \geqslant j$. We must show $\varphi_{j+1,n+1}$ is an isomorphism.

The simplicial identities give us

$$\forall i < j, \qquad \begin{array}{lcl} d_i s_j = s_{j-1} d_i & \Longrightarrow & s_j(N_j A_n) \subset N_j A_{n+1} \\ s_j s_i = s_i s_{j-1} & \Longrightarrow & s_j(D_j A_n) \subset D_j A_{n+1}, \end{array}$$

so the following commutes with exact rows

$$\begin{tikzcd}
 & & N_jA_n \arrow[dr, hook] \arrow[drr, "\varphi_{j,n}"] \arrow[dd, "s_j"] & & & \\
0 \arrow[r] & D_jA_n \arrow[rr] \arrow[dd, "s_j"] & & A_n \arrow[r] \arrow[dd, "s_j"] & \dfrac{A_n}{D_jA_n} \arrow[r] \arrow[dd, "s_j"] & 0 \\
 & & N_jA_{n+1} \arrow[dr, hook] \arrow[drr, "\varphi_{j,n+1}"] & & & \\
0 \arrow[r] & D_jA_{n+1} \arrow[rr] & & A_{n+1} \arrow[r] & \dfrac{A_{n+1}}{D_jA_{n+1}} \arrow[r] & 0
\end{tikzcd}.$$

Turning our attention to the square extending $\varphi_{j,n}$ to $\varphi_{j,n+1}$, we calculate

$$\operatorname{coker}\left(\frac{A_n}{D_jA_n} \xrightarrow{s_j} \frac{A_{n+1}}{D_jA_{n+1}}\right) \cong \frac{A_{n+1}}{D_jA_{n+1} + s_j(A_n)} \cong \frac{A_{n+1}}{D_{j+1}A_{n+1}},$$

giving an isomorphism of exact sequences[1]

$$\begin{tikzcd}
0 \arrow[r] & N_jA_n \arrow[r, "s_j"] \arrow[d, "\wr", "\varphi_{j,n}"'] & N_jA_{n+1} \arrow[r] \arrow[d, "\wr", "\varphi_{j,n+1}"'] & K \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \dfrac{A_n}{D_jA_n} \arrow[r, "s_j"] & \dfrac{A_{n+1}}{D_jA_{n+1}} \arrow[r] & \dfrac{A_{n+1}}{D_{j+1}A_{n+1}} \arrow[r] & 0,
\end{tikzcd}$$

where $K = \operatorname{coker}\left(N_jA_n \xrightarrow{s_j} N_jA_{n+1}\right)$. However,

$$\ker\left(N_jA_{n+1} \xrightarrow{d_j} N_jA_n\right) = N_{j+1}A_{n+1},$$

so the simplicial identity $d_js_j = 1$ gives us the splitting

$$\begin{tikzcd}
0 \arrow[r] & N_{j+1}A_{n+1} \arrow[r] & N_jA_{n+1} \arrow[r, "d_j"] & N_jA_n \arrow[l, bend left, dashed, "s_j"] \arrow[r] & 0,
\end{tikzcd}$$

and hence $N_{j+1}A_{n+1} \cong K \cong \dfrac{A_{n+1}}{D_{j+1}A_{n+1}}$. To see that this isomorphism is given by $\varphi_{j+1,n+1}$, we note that

$$\varphi_{j+1,n+1} = N_{j+1}A_{n+1} \hookrightarrow N_jA_{n+1} \xrightarrow{\varphi_{j,n+1}} \frac{A_{n+1}}{D_jA_{n+1}} \twoheadrightarrow \frac{A_{n+1}}{D_{j+1}A_{n+1}}$$

[1]The simplicial identity $d_js_j = 1$ implies s_j is a monomorphism and d_j is an epimorphism for a simplicial object in any category.

and observe that this composition appears in the following diagram,

$$
\begin{tikzcd}
 & & 0 \arrow[dd] & & 0 \arrow[dd] & \\
 & & & 0 \arrow[dd] & & \\
 & 0 \arrow[r] \arrow[dd] & N_jA_n \arrow[rr] \arrow[dr, "\varphi_{j,n}"] \arrow[dd, "s_j"] & & N_jA_n \arrow[r] \arrow[dd] & 0 \\
 & & & \frac{A_n}{D_jA_n} \arrow[dd] & & \\
0 \arrow[r] & N_{j+1}A_{n+1} \arrow[r, red] \arrow[dd] & N_jA_{n+1} \arrow[rr, "d_j"] \arrow[dr, red, "\varphi_{j,n+1}"] \arrow[dd] & & N_jA_n \arrow[r] \arrow[dd] & 0 \\
 & & & \frac{A_{n+1}}{D_jA_{n+1}} \arrow[dd, red] & & \\
0 \arrow[r] & N_{j+1}A_{n+1} \arrow[r] \arrow[dd] & K \arrow[rr] \arrow[dr, "\sim"] \arrow[dd] & & 0 & \\
 & & & \frac{A_{n+1}}{D_{j+1}A_{n+1}} \arrow[dd] & & \\
 & 0 & 0 & & & \\
 & & & 0 & &
\end{tikzcd}
$$

$\square$

3. Simplicial Objects from Chain Complexes

To recover the degenerate from the nondegenerate simplices in a simplicial set X, we notice that the codegeneracy maps define disjoint injections of the nondegenerate n-simplices into the $(n+1)$-simplices. In fact, the epimorphisms $[m] \twoheadrightarrow [n]$ define disjoint injections of the nondegenerate n-simplices into the m-simplices, i.e., the degenerate simplices are uniquely determined as the degenerate images of the nondegenerate simplices.

We can also recover the action of any map $\eta : [m] \to [n]$ from the nondegenerate maps. Let $x \in X_n$ and find its corresponding nondegenerate simplex $x' \in X_p$, where $\alpha^*(x') = x$ for some $\alpha : [n] \twoheadrightarrow [p]$. Taking the epi-mono factorization of this composition $\alpha\eta = \mu\varepsilon$ tells us $\mu^*(x')$ is the nondegenerate simplex associated to $\eta^*(x)$.

$$
\begin{tikzcd}
X_n \arrow[rrr, "\eta^*"] & & & X_m \\
 & x \arrow[r, mapsto] & \eta^*(x) & \\
 & x' \arrow[r, mapsto] \arrow[u] & \mu^*(x') \arrow[u] & \\
X_p \arrow[uuu, "\alpha^*"] \arrow[rrr, "\mu^*"'] & & & X_k \arrow[uuu, "\varepsilon^*"']
\end{tikzcd}
$$

Thus, on $\alpha^*(X_p) \subset X_n$, we can compute η^* purely in terms of its nondegenerate restriction μ^*.

We now apply this process to simplicial objects in $\mathcal{A}$.

Definition 3.1. Let $\Gamma : \mathrm{ch}\mathcal{A} \to \mathrm{s}\mathcal{A}$ be the functor defined by

$$\Gamma X_n = \bigoplus_{\alpha:[n]\twoheadrightarrow[p]} X_\alpha, \qquad \text{where } X_\alpha := X_p,$$

and for $\eta : [m] \to [n]$, η^* is defined for the component $\alpha : [n] \twoheadrightarrow [p]$ by

$$\begin{tikzcd} \bigoplus_{\alpha:[n]\twoheadrightarrow[p]} X_\alpha \arrow[r, "\eta^*"] & \bigoplus_{\beta:[n]\twoheadrightarrow[p]} X_\beta \\ X_\alpha \arrow[u, hook] \arrow[r, "\eta^*_\alpha"'] & X_\varepsilon \arrow[u, hook] \end{tikzcd} \qquad \text{where} \qquad \begin{tikzcd} {[m]} \arrow[r, "\eta"] \arrow[d, two heads, "\varepsilon"'] & {[n]} \arrow[d, two heads, "\alpha"] \\ {[k]} \arrow[r, hook, "\mu"'] & {[p]} \end{tikzcd}$$

is the epi-mono factorization of $\alpha\eta$, and $\eta^*_\alpha = \begin{cases} 1 & p = k \\ \partial^X_p & p = k+1, \mu = d^p \\ 0 & \text{otherwise} \end{cases}$.

Remark 3.2. If $\eta : [m] \twoheadrightarrow [n]$, then for every $\alpha : [n] \twoheadrightarrow [p]$, $\alpha\eta$ is epi, so η^* is the inclusion

$$\bigoplus_{\alpha:[n]\twoheadrightarrow[p]} X_\alpha \xrightarrow{\sim} \bigoplus_{\alpha\eta:[m]\twoheadrightarrow[p]} X_{\alpha\eta} \hookrightarrow \bigoplus_{\beta:[m]\twoheadrightarrow[p]} X_\beta$$

which implies

$$D\left(\bigoplus_{\alpha:[n]\twoheadrightarrow[p]} X_\alpha\right) = \bigoplus_{\substack{\alpha:[n]\twoheadrightarrow[p] \\ n>p}} X_\alpha$$

and hence

$$N\left(\bigoplus_{\alpha:[n]\twoheadrightarrow[p]} X_\alpha\right) = X_{1_{[n]}} = X_n.$$

4. The Dold-Kan Theorem

We now present the main theorem.

Theorem 4.1 (Dold-Kan). $\mathrm{ch}\mathcal{A} \underset{N}{\overset{\Gamma}{\rightleftarrows}} \mathrm{s}\mathcal{A}$ *is an equivalence of categories.*

Proof. ($N\Gamma \cong 1_{\mathrm{ch}\mathcal{A}}$) This follows from remark 3.2

($\Gamma N \cong 1_{\mathrm{s}\mathcal{A}}$) Define $\psi : \Gamma N \Rightarrow 1_{\mathrm{s}\mathcal{A}}$ on the α component of $\Gamma N A_n$ by

$$\begin{tikzcd} \bigoplus_{\alpha:[n]\twoheadrightarrow[p]} NA_\alpha \arrow[r, "\psi_{A,n}"] & A_n \\ NA_\alpha \arrow[u, hook] \arrow[r, hook] & A_p \arrow[u, "\alpha^*"'] \end{tikzcd}.$$

To see that ψ_A is a well defined morphism of simplicial objects, fix $\eta : [m] \to [n]$ and consider the restriction to the component NA_α of ΓNA_n for some $\alpha : [n] \twoheadrightarrow [p]$. If

$$\begin{tikzcd}
{[m]} \arrow[r, "\eta"] \arrow[d, two heads, "\varepsilon"'] & {[n]} \arrow[d, two heads, "\alpha"] \\
{[k]} \arrow[r, hook, "\mu"'] & {[p]}
\end{tikzcd}$$

is the epi-mono factorization of $\alpha\eta$, we must verify commutativity of the inner square in the following diagram

$$\begin{tikzcd}
\Gamma NA_n \arrow[rrr, "\psi_{A,n}"] \arrow[ddd, "\eta^*"'] & & & A_n \arrow[ddd, "\eta^*"] \\
& NA_\alpha \arrow[ul, hook] \arrow[r, hook] \arrow[d, "\eta^*_\alpha"'] & A_p \arrow[ur, hook, "\alpha^*"] \arrow[d, "\mu^*"] & \\
& NA_\varepsilon \arrow[dl, hook] \arrow[r, hook] & A_k \arrow[dr, hook, "\varepsilon^*"] & \\
\Gamma NA_m \arrow[rrr, "\psi_{A,m}"'] & & & A_m
\end{tikzcd} .$$

If $\mu = d^p$, then $\eta^*_\alpha = \partial^{NA}_p = d_p$, and if $\mu = 1_{[p]}$ then $\eta^*_\alpha = 1_{NA_p}$. In any other case, the simplicial identities guarantee μ^* can be factored as $\mu^* = \xi^* d_i$ where $i < p$, implying $\mu^*|_{NA_p} = 0$. Since $\eta^*_\alpha = 0$ in this case, the inner square commutes as desired.

Note that $\psi_{A,0}$ is an isomorphism. Fix $n > 0$ and suppose $\psi_{A,k}$ is an isomorphism for all $k < n$. By remark 3.2, $\psi_{A,n}$ restricts to an isomorphism

$$N\psi_{A,n} : N\Gamma NA_n \xrightarrow{\sim} NA_n$$

and for each j, commutativity of the square

$$\begin{tikzcd}
\Gamma NA_n \arrow[r, "\psi_{A,n}"] & A_n \\
\Gamma NA_{n-1} \arrow[u, hook, "s_j"] \arrow[r, "\psi_{A,n-1}"'] & A_{n-1} \arrow[u, hook, "s_j"']
\end{tikzcd}$$

implies $\psi_{A,n}$ restricts to an isomorphism

$$D\psi_{A,n} : D\Gamma NA_n = \sum_j s_j(\Gamma NA_{n-1}) \xrightarrow{\sim} \sum_j s_j(A_{n-1}) = DA_n.$$

Thus, by naturality of $NA_n \oplus DA_n \xrightarrow{\sim} A_n$,

$$\begin{tikzcd}
N\Gamma NA_n \oplus D\Gamma NA_n \arrow[r, "N\psi_{A,n} \oplus D\psi_{A,n}"] \arrow[d, "\wr"'] & DA_n \oplus NA_n \arrow[d, "\wr"] \\
\Gamma NA_n \arrow[r, "\psi_{A,n}"'] & A_n
\end{tikzcd}$$

commutes, hence $\psi_{A,n}$ is an isomorphism. □