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Classical Solutions for Diffusion-Induced Grain-Boundary Motion

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We prove existence and uniqueness of classical solutions for the motion of hypersurfaces driven by mean curvature and diffusion of a solute along the surface. This free boundary problem involves solving a coupled system of nonlinear partial differential equations.

1. INTRODUCTION

In this paper we study the motion of hypersurfaces driven by mean curvature and by diffusion of a solute along the surface. Let Γ_0 be a compact closed hypersurface in $\mathbb{R}^n$ which is the boundary of an open domain, and let $u_0\colon \Gamma_0 \to \mathbb{R}$ be a given function. Then we are looking for a family $\Gamma := \{\Gamma(t);\ t \geq 0\}$ of hypersurfaces and a family of functions $\{u(\cdot, t)\colon \Gamma(t) \to \mathbb{R};\ t \geq 0\}$ such that the system of equations holds

$$\begin{aligned} V &= -H_\Gamma - f(u), & \Gamma(0) &= \Gamma_0, \\ \frac{du}{dt} &= \Delta_\Gamma u - V H_\Gamma u + Vu + g(u), & u(0) &= u_0. \end{aligned} \tag{1.1}$$

Here $V(t)$ denotes the normal velocity of Γ at time t, while $H_{\Gamma(t)}$ and $\Delta_{\Gamma(t)}$ stand for the mean curvature and the Laplace–Beltrami of $\Gamma(t)$, respectively. The symbol $\frac{du}{dt}$ denotes the derivative of u along flow lines which are orthogonal to $\Gamma(t)$; the details are explained further below, see

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the definition in (2.6). We assume that

$$f, g \in C^{\infty}(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad f(0) = 0, \qquad g(0) = U.$$

In two dimensions, the interface $\Gamma(t)$ represents the boundary of a grain of a thin (essentially two-dimensional) poly-crystalline material with vapor on top (in the third dimension). The interface is driven by surface tension, and by diffusion, known as DIGM (diffusion induced grain motion). That is, the vapor in the third dimension is assumed to contain a certain solute which is absorbed by the interface and which is diffused along the interface. Furthermore, as the interface moves, some of the solute is deposited in the bulk through which the interface has passed. The chemical composition of the newly created crystal behind the advancing grain is different from that in front, because atoms of the solute have been deposited there. For this physical background we consider only convex curves, and we choose the signs so that a family of shrinking curves has negative normal velocity. A high concentration u of the solute in the interface increases the velocity, because the interface tries to reduce that concentration by depositing the solute in the regions it passes through. In addition, the stretching or shrinking of Γ during its motion induces a change in the concentration of the solute. All in all this results in the following terms: $V = -H_\Gamma$ is the usual motion by mean curvature that models motion driven purely by surface tension, and the term $f(u)$ results from the deposition effect (physically, $f(u) = u^2$ is considered reasonable [10]). As for the second equation, $\frac{du}{dt} = \Delta_\Gamma u$ describes diffusion on a manifold, $-VH_\Gamma u$ indicates the concentration change due to the change of the length of the interface, Vu describes the reduction of the solute due to deposition, and $g(u)$ results from the absorption of the solute from the vapor. Physically, $g(u) = U - u$ is meaningful, where U is the concentration of the solute in the vapor.

It should be observed that (1.1) reduces to the well-known mean curvature flow

$$V = -H_\Gamma, \qquad \Gamma(0) = \Gamma_0, \tag{1.2}$$

if $u_0 = 0$ and $U = 0$, because $u \equiv 0$ then solves the second equation of (1.1).

DIGM is known to be an important component of many complicated diffusion processes in which there are moving grain boundaries; see [5] and the references cited therein. In this type of phenomenon, the free energy of the system can be reduced by the incorporation of some of the solute into one or both of the grains separated by the grain boundary. In the DIGM mechanism, this transfer is accomplished by the disintegration of one grain and the simultaneous building up of the adjacent grain, the solute being added during the build-up process. This results in the migration of the grain boundary [10]. The possibility of reducing the free energy this way does not automatically imply that migration actually takes place; mechanisms for this to happen have been proposed, including the one in [5].

In [5], a thermodynamically consistent phase-field model for DIGM is suggested which incorporates many attributes of a real grain boundary: a thin movable zone in which diffusion is rapid, located between two much larger regions in which diffusion is negligible. This model has two phase fields, one being the concentration of the solute, and the other one being an order parameter which distinguishes the two crystal grains by the values $+1$ and -1 and which takes intermediate values in the grain boundary.

In this paper we consider a sharp interface model for DIGM and we prove existence and uniqueness of local classical solutions. This model can be derived from the phase-field model by an asymptotic analysis, as will be shown in [4].

Given $T > 0$, let $\Gamma := \{\Gamma(t);\ t \in [0,T)\}$ be a family of closed compact hypersurfaces in $\mathbb{R}^n$. Then we set

$$\bar{\mathscr{M}} := \bigcup_{t\in[0,T)} \Gamma(t) \times \{t\}, \qquad \mathscr{M} := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}. \tag{1.3}$$

Let u be a real-valued function on $\bar{\mathscr{M}}$. Then we call (Γ, u) a classical solution of (1.1) on $[0,T)$ if $\mathscr{M}$ is an n-dimensional manifold of class C^2 in $\mathbb{R}^{n+1}$, if $u|_{\mathscr{M}} \in C^2(\mathscr{M})$, and if the pair (Γ, u) satisfies System (1.1) for $t > 0$. Moreover, we ask that $\bar{\mathscr{M}}$ is a C^1-manifold with boundary $\bar{\mathscr{M}} \cap (\mathbb{R}^n \times \{0\})$, that $u \in C^1(\bar{\mathscr{M}})$, and that $\bar{\mathscr{M}} \cap (\mathbb{R}^n \times \{0\}) = \Gamma_0 \times \{0\} \equiv \Gamma_0$ and $u|_{\Gamma_0} = u_0$.

The basic existence and uniqueness result for System (1.1) is contained in the following result.

THEOREM 1.1. *Let $\beta \in (0,1)$ be given and suppose that $\Gamma_0 \in C^{2+\beta}$ and that $u_0 \in C^{2+\beta}(\Gamma_0)$. Then System* (1.1) *has a classical solution (Γ, u) on $[0,T)$ for some $T > 0$. The solution is unique in the class* (4.6).

We mention that the solutions of System (1.1) inherit spatial $C^{2+\alpha}$-regularity for $\alpha < \beta$ from (u_0, Γ_0), see the proof of Theorem 1.1 at the end of this paper. It follows from Theorem 4.3 that there is no loss of regularity if (Γ_0, u_0) satisfy an appropriate additional assumption.

System (1.1) constitutes a nonlinear coupled system of equations. A detailed analysis discloses that System (1.1) is, in fact, fully nonlinear, caused by the term $VH_\Gamma u$. In order to investigate this system we represent the moving hypersurface $\Gamma(t)$ as a graph over a fixed reference manifold Σ

and then transform (1.1) to an evolution equation over Σ. This leads to a fully nonlinear system which is shown to be parabolic (in the sense that the linearization generates an analytic semigroup on an appropriate function space). We then use results on maximal regularity to compensate for the loss of derivatives caused by the fully nonlinear character. A similar approach has been successful in the study of various equations describing the motion of hypersurfaces driven by mean curvature, see [7–9] for instance. However, we mention that the equations in [7–9] carry, in contrast to (1.1), a quasi-linear parabolic structure.

It is well known that solutions of the mean curvature flow (1.2) remain strictly convex if Γ_0 is strictly convex, and that $\Gamma(t)$ shrinks to a point in finite time [11, 13]. Moreover, embedded curves in the plane always become convex before they shrink to a point [12]. We do not know if similar properties hold true for System (1.1).

2. MOTION OF THE INTERFACE

In this section we introduce the mathematical setting in order to reformulate (1.1) as an evolution equation over a fixed reference manifold.

Let Σ be a smooth compact closed hypersurface in $\mathbb{R}^n$, and assume that Γ_0 is close in a C^1 sense to this fixed reference manifold Σ. Let ν be the unit normal field on Σ. We choose $a > 0$ such that

$$X\colon \Sigma \times (-a, a) \to \mathbb{R}^n, \qquad X(s, r) := s + r\nu(s)$$

is a smooth diffeomorphism onto its image $\mathscr{R} := \operatorname{im}(X)$, that is,

$$X \in \operatorname{Diff}^\infty(\Sigma \times (-a, a), \mathscr{R}).$$

This can be done by taking $a > 0$ sufficiently small so that Σ has a tubular neighborhood of radius a. It is convenient to decompose the inverse of X into $X^{-1} = (S, \Lambda)$, where

$$S \in C^\infty(\mathscr{R}, \Sigma) \quad \text{and} \quad \Lambda \in C^\infty(\mathscr{R}, (-a, a)).$$

$S(x)$ is the nearest point on Σ to $x \in \mathscr{R}$, and $\Lambda(x)$ is the signed distance from x to Σ, that is, to $S(x)$. Moreover, $\mathscr{R}$ consists of those points in $\mathbb{R}^n$ with distance less than a to Σ.

Let $T > 0$ be a fixed number. In the sequel we assume that $\Gamma := \{\Gamma(t), t \in [0, T)\}$ is a family of graphs in normal direction over Σ. To be precise, we ask that there is a function $\rho\colon \Sigma \times [0, T) \to (-a, a)$ such that

$$\Gamma(t) = \operatorname{im}([s \mapsto X(s, \rho(s, t))]), \qquad t \in [0, T).$$

$\Gamma(t)$ can then also be described as the zero-level set of the function

$$\Phi_\rho\colon \mathscr{R} \times [0,T) \to \mathbb{R}, \qquad \Phi_\rho(x,t) := \Lambda(x) - \rho(S(x),t), \quad (2.1)$$

and one has $\Gamma(t) = \Phi_\rho(\cdot,t)^{-1}(0)$ for any fixed $t \in [0,T)$. Hence, the unit normal field $N(x,t)$ on $\Gamma(t)$ at x can be expressed as

$$N(x,t) = \frac{\nabla_x \Phi_\rho(x,t)}{|\nabla_x \Phi_\rho(x,t)|}, \quad (2.2)$$

and the normal velocity V of Γ at time t and at the point $x = X(s, \rho(s,t))$ is given by

$$V(x,t) = \frac{\partial_t \rho(s,t)}{|\nabla_x \Phi_\rho(x,t)|}. \quad (2.3)$$

We can now explain the precise meaning of the derivative $\frac{du}{dt}(x,t)$ for $x \in \Gamma(t)$. Given $x \in \Gamma(t)$, let $\{z(\tau,x) \in \mathbb{R}^n;\ \tau \in (-\varepsilon,\varepsilon)\}$ be a flow line through x such that

$$z(\tau,x) \in \Gamma(t+\tau), \quad \dot{z}(\tau) = (VN)(z(\tau), t+\tau),$$
$$\tau \in (-\varepsilon,\varepsilon),\ z(0) = x. \quad (2.4)$$

The existence of a unique trajectory $\{z(\tau,x) \in \mathbb{R}^n;\ \tau \in (-\varepsilon,\varepsilon)\}$ with the above properties is established in the next lemma.

LEMMA 2.1. *Suppose* $\rho \in C^2(\Sigma \times (0,T))$ *and let* $\Gamma(t) := \Phi_\rho(\cdot,t)^{-1}(0)$ *for* t *in* $(0,T)$. *Then for every* $x \in \Gamma(t)$ *there exist an* $\varepsilon > 0$ *and a unique solution* $z(\cdot,x) \in C^1((-\varepsilon,\varepsilon),\mathbb{R}^n)$ *of* (2.4).

Proof. Note that (2.4) is equivalent to the ordinary differential equation

$$(\dot{z},\dot{t}) = ((VN)(z,t),1), \qquad (z(0),t(0)) = (x,t), \quad (2.5)$$

on the manifold $\mathscr{M} = \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\}$. We show that

$$((VN)(x,t),1) \in T_{(x,t)}(\mathscr{M}) \quad \text{for any } (x,t) \in \mathscr{M}.$$

For this let $\Psi_\rho := \Psi_\rho|_{\mathscr{R}\times(0,T)}$ and observe that $\mathscr{M} = \Psi_\rho^{-1}(0)$, so that the vector

$$\left(\nabla_x \Phi_\rho(x,t), -\partial_t \rho(S(x),t)\right)$$

is orthogonal to $\mathscr{M}$ at $(x,t) \in \mathscr{M}$. Using the definition of Φ_ρ it can easily be seen that $\partial_\nu \Phi_\rho = 1$, and hence the vector displayed above is nonzero. By (2.2) and (2.3) we have

$$\Big(((VN)(x,t),1) \,\Big|\, \big(\nabla_x \Phi_\rho(x,t), -\partial_t \rho(S(x),t)\big)\Big) = 0, \qquad (x,t) \in \mathscr{M},$$

showing that $((VN)(x,t),1)$ is a tangential vector to $\mathscr{M}$ at (x,t), because $\mathscr{M}$ has codimension 1 in $\mathscr{R} \times (0,T)$. We can now conclude that there is an $\varepsilon > 0$ such that (2.5) has a unique solution

$$[\tau \mapsto (z(\tau,x), t+\tau)] \in C^1((-\varepsilon,\varepsilon),\mathscr{M}).$$

It follows that $[\tau \mapsto z(\tau,x)] \in C^1((-\varepsilon,\varepsilon),\mathbb{R}^n)$ is the unique solution of (2.4). ∎

Let $x \in \Gamma(t)$ for $t \in (0,T)$ be given. Then by definition we have

$$\frac{du}{dt}(x,t) := \frac{d}{d\tau} u(z(\tau,x), t+\tau)\Big|_{\tau=0}. \tag{2.6}$$

Formally one obtains

$$\frac{du}{dt}(x,t) = \big(\nabla_x u(x,t) | N(x,t)\big) V(x,t) + \partial_t u(x,t). \tag{2.7}$$

This equation is well known in continuum mechanics, but is here only formally correct because $u(\cdot,t)$ is defined on $\Gamma(t)$, but not on all of $\mathbb{R}^n$. For this reason we introduce the pull-back function v of u,

$$v\colon \Sigma \times [0,T) \to \mathbb{R}, \qquad v(s,t) := u(X(s,\rho(s,t)),t). \tag{2.8}$$

Using the mapping S introduced above one may also write $u(x,t) = v(S(x),t)$ for $x \in \Gamma(t)$. It follows from (2.4) and (2.6), and with $s := S(x)$, that

$$\begin{aligned} \frac{du}{dt}(x,t) &= \frac{d}{d\tau} v(S(z(\tau,x)), t+\tau)\Big|_{\tau=0} \\ &= \big(\nabla_x v(S(x),t) | N(x,t)\big) V(x,t) + \frac{dv}{dt}(s,t). \end{aligned}$$

This is exactly what one obtains formally with a change of variables from (2.7). We remark that $\frac{du}{dt}$ is so far only defined in (2.6) for $t \in (0,T)$. Therefore we set

$$\frac{du}{dt}(x,t) := \big(\nabla_x v(S(x),t) | N(x,t)\big) V(x,t) + \frac{dv}{dt}(s,t), \tag{2.9}$$

for $x \in \Gamma(t)$ and $t \in [0, T)$, where $s = S(x)$. For further use we introduce the functions

$$\begin{aligned} L(\rho)(s,t) &:= |\nabla_x \Phi_\rho(x,t)|\big|_{x = X(s,\, \rho(s,t))}, \\ I(\rho, v)(s,t) &:= \big(\nabla_x v(S(x),t) \mid N(x,t)\big)\big|_{x = X(s,\, \rho(s,t))}, \end{aligned} \tag{2.10}$$

for $(s,t) \in \Sigma \times [0,T)$. It follows that

$$\frac{du}{dt}(x,t)\bigg|_{x = X(s,\, \rho(s,t))} = \frac{dv}{dt}(s,t) + I(\rho, v)(s,t) V(x,t)\big|_{x = X(s,\, \rho(s,t))}. \tag{2.11}$$

3. THE TRANSFORMED EQUATIONS

Given an open set $U \subset \mathbb{R}^n$, let $h^s(U)$ denote the little Hölder space of order $s > 0$, that is, the closure of $BUC^\infty(U)$ in $BUC^s(U)$, the latter space being the Banach space of all bounded and uniformly Hölder continuous functions of order s. If Σ is a (sufficiently) smooth submanifold of $\mathbb{R}^n$ then the spaces $h^s(\Sigma)$ are defined by means of a smooth atlas for Σ. It is known that $BUC^t(\Sigma)$ is continuously embedded in $h^s(\Sigma)$ whenever $t > s$. Moreover, the little Hölder spaces have the interpolation property

$$\big(h^s(\Sigma), h^t(\Sigma)\big)_\theta = h^{(1-\theta)s + \theta t}(\Sigma), \qquad \theta \in (0,1), \tag{3.1}$$

whenever $s, t, (1-\theta)s + \theta t \in \mathbb{R}^+ \setminus \mathbb{N}$, where $(\cdot,\cdot)_\theta$ denotes the continuous interpolation method of DaPrato and Grisvard [6], see also [2, 3, 14].

In the following we fix $t \in [0, T)$ and drop it in our notation. Moreover, we fix $0 < \alpha < 1$ and define

$$U := \big\{\rho \in h^{1+\alpha}(\Sigma);\ \|\rho\|_{C(\Sigma)} < a\big\}, \qquad \mathbb{U} := U \times h^{1+\alpha}(\Sigma). \tag{3.2}$$

Given $\rho \in h^{2+\alpha}(\Sigma) \cap U$, we introduce the mapping

$$\theta_\rho \colon \Sigma \to \mathbb{R}^n, \qquad \theta_\rho(s) := X(s, \rho(s)) \quad \text{for } s \in \Sigma,\ \rho \in U.$$

It follows that θ_ρ is a well-defined $(2+\alpha)$-diffeomorphism from Σ onto $\Gamma_\rho := \operatorname{im}(\theta_\rho)$. We can, therefore, define the pull-back operator,

$$\theta_\rho^* u := u \circ \theta_\rho \quad \text{for } u \in C(\Gamma_\rho),$$

and the push-forward operator,

$$\theta_*^\rho v := v \circ \theta_\rho^{-1} \quad \text{for } v \in C(\Sigma),$$

induced by θ_ρ. Let Δ_{Γ_ρ} and H_{Γ_ρ} be the Laplace–Beltrami operator and the mean curvature, respectively, of Γ_ρ. Then we set

$$\Delta_\rho := \theta_\rho^* \Delta_{\Gamma_\rho} \theta_*^\rho, \qquad H(\rho) := \theta_\rho^* H_{\Gamma_\rho}.$$

We will now investigate the structure of the transformed operators Δ_ρ and $H(\rho)$. We begin with the transformed mean curvature operator $H(\rho)$.

LEMMA 3.1. *There exist functions*

$$P \in C^\infty\big(U, \mathscr{L}\big(h^{2+\alpha}(\Sigma), h^\alpha(\Sigma)\big)\big) \quad \textit{and} \quad K \in C^\infty(U, h^\alpha(\Sigma))$$

such that

$$H(\rho) = P(\rho)\rho + K(\rho) \quad \textit{for } \rho \in h^{2+\alpha}(\Sigma) \cap U.$$

Proof. This is quoted from [8, Lemma 3.1]. ∎

Note that Δ_ρ is the Laplace–Beltrami operator on $(\Sigma, \theta_\rho^* \eta)$. Here η is the Euclidean metric of $\mathbb{R}^n$ restricted to the manifold Γ_ρ, and $\theta_\rho^* \eta$ denotes the Riemannian metric that is induced by θ_ρ on the manifold Σ. To simplify the notation we set $\sigma(\rho) := \theta_\rho^* \eta$. Let $\sigma_{jk}(\rho)$ be the components of $\sigma(\rho)$ in local coordinates and let $\sigma^{jk}(\rho)$ be the entries of the inverse matrix of $[\sigma_{jk}(\rho)]$. Finally, let $\gamma_{jk}^i(\rho)$ denote the Christoffel symbols of $\sigma(\rho)$. Using local coordinates, one has

$$\Delta_\rho = \sigma^{jk}(\rho)\big(\partial_j \partial_k - \gamma_{jk}^i(\rho)\partial_i\big), \qquad \rho \in h^{2+\alpha}(\Sigma) \cap U. \tag{3.3}$$

The components $\sigma_{jk}(\rho)$ are given by $\sigma_{jk}(\rho) := (\partial_j \theta_\rho | \partial_k \theta_\rho)$. More precisely, let $\psi\colon \mathbb{R}^{n-1} \to \Sigma$ be some coordinate chart of Σ. Then

$$\sigma_{jk}(\rho)(x) := \left(\frac{\partial}{\partial x_j} \theta_\rho(\psi(x)) \,\middle|\, \frac{\partial}{\partial x_k} \theta_\rho(\psi(x)) \right)$$

for $x \in \mathbb{R}^{n-1}$. However, we shall prefer the shorter notation without mentioning ψ explicitly. With this in mind, the Christoffel symbols are defined by

$$\gamma_{jk}^i(\rho) := \frac{1}{2}\sigma^{im}(\rho)\big(\partial_k \sigma_{jm}(\rho) + \partial_j \sigma_{km}(\rho) - \partial_m \sigma_{jk}(\rho)\big),$$

$$\rho \in h^{2+\alpha}(\Sigma) \cap U. \tag{3.4}$$

Because $\theta_\rho(s) = s + \rho(s)\nu(s)$, it is easy to see that the (locally defined) mappings

$$\left[\rho \mapsto \left(\sigma_{jk}(\rho), \sigma^{jk}(\rho), \gamma^i_{jk}(\rho)\right)\right] : h^{2+\alpha}(\Sigma) \cap U \to \left(h^\alpha(\mathbb{R}^{n-1})\right)^3 \quad (3.5)$$

are smooth for each $i, j, k \in \{1, \ldots, n-1\}$. For further use we also note that

$$\sigma_{jk}(\rho) = \partial_j \rho\, \partial_k \rho + w_{jk}(\rho), \quad (3.6)$$

where the functions $w_{jk}(\rho)$ do not involve derivatives of ρ. In the following lemma we show that Δ_ρ depends smoothly on ρ.

LEMMA 3.2. *There exists a function* $C \in C^\infty(h^{2+\alpha}(\Sigma) \cap U, \mathscr{L}(h^{2+\alpha}(\Sigma), h^\alpha(\Sigma)))$ *such that*

$$\Delta_\rho v = C(\rho) v \quad \textit{for } \rho \in h^{2+\alpha}(\Sigma) \cap U, v \in h^{2+\alpha}(\Sigma).$$

Proof. Note that Δ_ρ is globally defined and that it is enough to verify the statement locally. Let $C(\rho)$ be defined in local coordinates by

$$C(\rho) := \sigma^{jk}(\rho)\left(\partial_j \partial_k - \gamma^i_{jk}(\rho)\, \partial_i\right).$$

Lemma 3.2 now follows from (3.3)–(3.5) by a localization argument. ∎

We remark that the mapping $[(\rho, v) \mapsto C(\rho)v]$ actually has a quasi-linear structure. This follows from the observation that second-order derivatives in ρ can only occur together with first-order derivatives of v, and vice versa. Because we do not need this information we refrain from writing down the details.

Our next result concerns smoothness properties of the mappings I and L introduced in (2.10), and the substitution operators induced by the functions f and g.

LEMMA 3.3. (a) $[\rho \mapsto L(\rho)] \in C^\infty(U, h^\alpha(\Sigma))$,
(b) $[(\rho, v) \to I(\rho, v)] \in C^\infty(\mathbb{U}, h^\alpha(\Sigma))$,
(c) $[v \to (f(v), g(v))] \in C^\infty(h^{2+\alpha}(\Sigma), (h^{2+\alpha}(\Sigma))^2)$.

Proof. Let η be the restriction of the Euclidean metric on $T(\mathscr{R})$ and let $g := X^*\eta$ be the pull-back metric on $T(\Sigma \times (-a, a))$. Because the mapping X is a translation in the normal direction of Σ, it follows that the metric g splits along the fibers of $\Sigma \times (-a, a)$, that is, $g = w(r) + dr \otimes dr$. Here r denotes the coordinate in the normal direction of Σ, and $w(r)$ is a metric on $T(\Sigma \times \{r\}) \equiv T(\Sigma)$. Given $\rho \in U$, we set

$$g(\rho) = w(\rho) + dr \otimes dr := g|_{(s,\, \rho(s))} \quad \text{on } T_{(s,\, \rho(s))}(\Sigma \times (-a, a)).$$

It follows that $w(\rho)$ induces a metric on $T(\Sigma)$. Let $w_{jk}(\rho)$ be the components of $w(\rho)$ in local coordinates and note that the two metrics $\sigma(\rho)$ and $w(\rho)$ on $T(\Sigma)$ are connected by (3.6). Let $w^{jk}(\rho)$ be the entries of the inverse matrix of $[w_{jk}(\rho)]$.

(a) Let $\hat{\Phi}_\rho := X^*\Phi_\rho$ be the pull-back of the function Φ_ρ. Then

$$L(\rho) = \sqrt{g(\rho)\left(\nabla_\Xi \hat{\Phi}_\rho, \nabla_\Xi \hat{\Phi}_\rho\right)},$$

where

$$\Xi := (\Sigma \times (-a, a), g).$$

Because $\hat{\Phi}_\rho(r, s) = r - \rho(s)$ it is easy to see that $L(\rho)$ can be expressed in local coordinates by

$$L(\rho) = \sqrt{1 + w^{lm}(\rho)\, \partial_l \rho\, \partial_m \rho}\,.$$

We can now conclude that the mapping $[\rho \mapsto L(\rho)]$ is smooth.

(b) It follows from (2.2) that

$$L(\rho) I(\rho, v)(s) = -(\nabla_x v(S(x)) | \nabla_x \rho(S(x)))\Big|_{x = X(s, \rho(s))},$$

because the vector $\nabla_x \Lambda(x)$ and $\nabla_x v(S(x))$ are orthogonal with respect to the Euclidean inner product. We obtain, using local coordinates, that $L(\rho) I(\rho, v) = -w^{lm}(\rho) \partial_l \rho\, \partial_m v$ and this shows that the mapping $[(\rho, v) \mapsto I(\rho, v)]$ is smooth, because $L(\rho)$ is a smooth strictly positive function.

(c) The statement in (c) follows from known properties of substitution operators on little Hölder spaces. ∎

4. EXISTENCE AND UNIQUENESS OF SOLUTIONS

We will now investigate the transformed system of equations

$$\begin{pmatrix} \dfrac{d\rho}{dt} \\ \dfrac{dv}{dt} \end{pmatrix} = \underbrace{\begin{pmatrix} -L(\rho)H(\rho) - L(\rho)f(v) \\ \Delta_\rho v + (I(\rho, v) + H(\rho)v - v)(H(\rho) + f(v)) + g(v) \end{pmatrix}}_{=: F(\rho, v)}, \tag{4.1}$$

$$(\rho(0), v(0)) = (\rho_0, v_0).$$

In the following, we call (ρ, v) a classical solution of (4.1) on $[0, T)$ if

$$(\rho, v) \in C^1(\Sigma \times [0, T), \mathbb{R}^2) \cap C^2(\Sigma \times (0, T), \mathbb{R}^2), \tag{4.2}$$

and if (ρ, v) satisfies System (4.1) pointwise for $t > 0$ and assumes the initial conditions.

LEMMA 4.1. *Systems* (1.1) *and* (4.1) *are equivalent*: *If* (1.1) *has a classical solution on* $[0, T)$, *then* (4.1) *also has a classical solution on* $[0, T)$, *and vice versa*.

Proof. (a) Let us first assume that (Γ, u) is a classical solution of (1.1) on $[0, T)$. Let Σ be a smooth manifold close to Γ_0 in the C^1 sense, as considered in Section 2. If $T > 0$ is small enough, $\Gamma(t) \subset \mathscr{R}$ for all $t \in [0, T)$, and the normal of $\Gamma(t)$ is close to the normal of Γ_0, so that $\Gamma(t)$ can also be represented as a graph over Σ. This is where we need the continuity of the solution in the C^1 sense up to $t = 0$. We find a unique function

$$\rho \in C^1(\Sigma \times [0, T)) \cap C^2(\Sigma \times (0, T))$$

such that $\Gamma(t) = \mathrm{im}([s \mapsto X(s, \rho(s, t))])$. Let v be defined as in (2.8). Then

$$v \in C^1(\Sigma \times [0, T)) \cap C^2(\Sigma \times (0, T))$$

and it follows from our considerations in Sections 2 and 3 that the pair (ρ, v) is a classical solution of System (4.1) on $[0, T)$.

(b) Conversely, assume that (ρ, v) is a classical solution of (4.1) on $[0, T)$. Let $\Gamma(t) := \mathrm{im}([s \mapsto X(s, \rho(s, t))])$ for $t \in [0, T)$ and let $\overline{\mathscr{M}}$ and $\mathscr{M}$ be defined as in (1.3). Then $\overline{\mathscr{M}}$ and $\mathscr{M}$ satisfy the regularity assumptions stated in the introduction, and so does the function u given by

$$u: \overline{\mathscr{M}} \to \mathbb{R}, \qquad u(x, t) := v(S(x), t), \qquad (x, t) \in \overline{\mathscr{M}}.$$

Our considerations in Sections 2 and 3 show, once more, that the pair (Γ, u) is a classical solution of (1.1) on $[0, T)$. ∎

Let E_0 and E_1 be Banach spaces such that E_1 is densely injected in E_0 and let $\mathscr{H}(E_1, E_0)$ denote the set of all $A \in \mathscr{L}(E_1, E_0)$ such that $-A$ is the generator of a strongly continuous analytic semigroup on E_0. It is known that $\mathscr{H}(E_1, E_0)$ is an open subset of $\mathscr{L}(E_1, E_0)$ which is given the relative topology of $\mathscr{L}(E_1, E_0)$. In the sequel we use the following Banach spaces:

$$E_1 := h^{2+\alpha}(\Sigma) \times h^{2+\alpha}(\Sigma), \qquad E_0 := h^{\alpha}(\Sigma) \times h^{\alpha}(\Sigma). \tag{4.3}$$

We now show that the evolution equation (4.1) is parabolic in the sense that the linearization of the mapping F generates a strongly continuous analytic semigroup on E_0. For further use we set $\mathbb{V} := E_1 \cap \mathbb{U}$.

PROPOSITION 4.2. *The mapping* $[(\rho, v) \mapsto F(\rho, v)]\colon \mathbb{V} \to E_0$ *is* C^∞ *and the Fréchet derivative* $F'(\rho_0, v_0)$ *of* F *at* (ρ_0, v_0) *satisfies* $-F'(\rho_0, v_0) \in \mathscr{H}(E_1, E_0)$ *for each* $(\rho_0, v_0) \in \mathbb{V}$.

Proof. It follows from Lemmas 3.1–3.3 and from the fact that $h^\alpha(\Sigma)$ is a multiplication algebra, that the mapping $F\colon \mathbb{V} \to E_0$ is smooth. The same references also show that the Fréchet derivative $F'(\rho_0, v_0)$ can be written as

$$F'(\rho_0, v_0) = A(\rho_0, v_0) + B(\rho_0, v_0), \qquad (\rho_0, v_0) \in \mathbb{V}, \qquad (4.4)$$

with $A(\rho_0, v_0) \in \mathscr{L}(E_1, E_0)$ and $B(\rho_0, v_0) \in \mathscr{L}(h^{1+\alpha}(\Sigma) \times h^{1+\alpha}(\Sigma),\, E_0)$, where

$$A(\rho_0, v_0) = \begin{pmatrix} -L(\rho_0)P(\rho_0) & 0 \\ v_0 C'(\rho_0) + \big(I(\rho_0, v_0) + 2H(\rho_0)v_0 + f(v_0)v_0 - v_0\big)P(\rho_0) & \Delta_{\rho_0} \end{pmatrix},$$

with $C'(\rho_0)$ the Fréchet derivative of C at ρ_0. It follows from the results in [7, Sect. 2] and from [2, Theorem I.1.6.1] that $A(\rho_0, v_0)$ generates a strongly continuous analytic semigroup on E_0, that is,

$$-A(\rho_0, v_0) \in \mathscr{H}(E_1, E_0) \quad \text{for each } (\rho_0, v_0) \in \mathbb{V}.$$

Moreover, (3.1) and [2, Proposition I.2.3.3] show that $h^{1+\alpha}(\Sigma) \times h^{1+\alpha}(\Sigma) = (E_0, E_1)_{1/2}$. A well-known perturbation result for generators of analytic semigroups now implies that $-F'(\rho_0, v_0) \in \mathscr{H}(E_1, E_0)$. ∎

It should be observed that the term $(H(\rho))^2$ appearing in the second line of (4.1) involves the product of second-order derivatives of the function ρ. This implies that System (4.1) has to be viewed as a fully nonlinear parabolic evolution equation. If the term $VH_\Gamma u$ was missing in (1.1), then it could be shown that the resulting system corresponding to (4.1) carried a quasi-linear parabolic structure, in which case we could use the theory of quasi-linear parabolic systems developed by Amann [1, 2].

To investigate System (4.1) we appeal to the theory of fully nonlinear evolution equations of parabolic type relying on optimal regularity results in the sense of DaPrato and Grisvard [6], see also [2, 3, 14].

THEOREM 4.3. *Given any* $w_0 := (\rho_0, v_0) \in \mathbb{V}$ *there exists a positive number* $T = T(w_0)$ *such that evolution equation* (4.1) *has a unique maximal* *classical solution*

$$(\rho(\cdot, w_0), v(\cdot, w_0)) \in C([0,T), \mathbb{V}) \cap C^1([0,T), E_0) \\ \cap C^{2+\alpha}(\Sigma \times (0,T), \mathbb{R}^2).$$

The mapping $[w_0 \mapsto (\rho(\cdot, w_0), v(\cdot, w_0))]$ *defines a smooth semiflow on* $\mathbb{V}$.

Proof. Let $I := [0, \tau]$ be given and let

$$\mathbb{E}_0 := C(I, E_0), \quad \mathbb{E}_1 := C(I, E_1) \cap C^1(I, E_0), \quad \mathbb{D}_1 := \mathbb{E}_1 \cap C(I, \mathbb{V}).$$

Using the fact that the little Hölder spaces are invariant under the continuous interpolation method (see (3.1)), the interpolation result [2, Proposition I.2.3.3] for product spaces, Proposition 4.2, and [2, Theorem III.3.4.1] with $\mu = 0$, it is not difficult to verify that $(\mathbb{E}_0, \mathbb{E}_1)$ is a pair of maximal regularity for $-F'(\rho_0, v_0)$, that is,

$$\left(\frac{d}{dt} - F'(\rho_0, v_0), \gamma\right) \in \operatorname{Isom}(\mathbb{E}_1, \mathbb{E}_0 \times E_1), \qquad (\rho_0, v_0) \in \mathbb{V}, \quad (4.5)$$

where $\gamma u := u(0)$ denotes the trace of u for $u \in \mathbb{E}_1$. Equation (4.5) shows that the linearized problem

$$\left(\frac{d}{dt}u - F'(\rho_0, v_0)u, \gamma u\right) = (f, u_0)$$

has for each $(f, u_0) \in \mathbb{E}_0 \times E_1$ a unique solution $u = u(f, u_0) \in \mathbb{E}_1$ which has the best possible regularity. Using the maximal regularity result (4.5) one proves that the mapping G,

$$G(\rho, v)(t) := e^{t\mathbb{A}}(\rho_0, v_0) \\ + \int_0^t e^{(t-s)\mathbb{A}}(F(\rho(s), v(s)) - \mathbb{A}(\rho(s), v(s)))\, ds,$$

with $\mathbb{A} := F'(\rho_0, v_0)$, maps $\mathbb{D}_1$ into itself and that G has a unique fixed point $(\rho(\cdot, w_0), v(\cdot, w_0))$ in $\mathbb{D}_1$, provided τ is small enough, see [3, Theorem 2.7] or [14, Theorem 8.4.1] for more details. A standard continuation argument then yields a unique maximal solution

$$(\rho(\cdot, w_0), v(\cdot, w_0)) \in C([0,T), \mathbb{V}) \cap C^1([0,T), E_0). \qquad (4.6)$$

It follows from [3, Corollary 2.10] that, in addition, the solution is smooth in t for $t > 0$, i.e.,

$$\left(\rho(\cdot, w_0), v(\cdot, w_0)\right) \in C^{\infty}\left((0, T), E_1\right). \tag{4.7}$$

We conclude from (4.6) and (4.7) that the solution satisfies the regularity properties stated in the theorem. Finally, [3, Corollary 2.9] also shows that the mapping $[w_0 \mapsto (\rho(\cdot, w_0), v(\cdot, w_0))]$ governs a smooth semiflow on $\mathbb{V}$. ▮

Proof of Theorem 1.1. Let Γ_0 be a given compact, closed $C^{2+\beta}$-manifold in $\mathbb{R}^n$. As in Section 2 we find a smooth reference manifold Σ and a function $\rho_0 \in C^{2+\beta}(\Sigma) \cap U$ such that

$$\Gamma_0 = \operatorname{im}\left(\left[s \mapsto X(s, \rho_0(s))\right]\right).$$

Because $C^{2+\beta}(\Sigma) \subset h^{2+\alpha}(\Sigma)$ for $\alpha \in (0, \beta)$ we also have that $\rho_0 \in h^{2+\alpha}(\Sigma) \cap U$. Given $u_0 \in C^{2+\beta}(\Gamma_0)$, let $v_0\colon \Sigma \to \mathbb{R}$ be defined by $v_0(s) := u_0(X(s, \rho_0(s)))$ for $s \in \Sigma$. We can conclude that $v_0 \in h^{2+\alpha}(\Sigma)$. Theorem 4.3 yields the existence of a unique solution,

$$\begin{aligned}\left(\rho(\cdot, w_0), v(\cdot, w_0)\right) \in C\left([0, T), \mathbb{V}\right) &\cap C^1\left([0, T), E_0\right)\\ &\cap C^{2+\alpha}\left(\Sigma \times (0, T)\right),\end{aligned}$$

for System (4.1), where we have set $w_0 = (\rho_0, v_0)$. Clearly, this solution also satisfies the regularity assumptions in (4.2). Lemma 4.1 then shows that (1.1) has a classical solution on $[0, T)$. The solution is unique in the class (4.6) as follows from Theorem 4.3, and the proof of Theorem 1.1 is now completed. ▮

Remark 4.4. Once the existence of unique classical solutions to the fully nonlinear evolution equation (4.1) is guaranteed, one can hope to show that solutions actually enjoy better regularity properties than stated in Theorem 4.3. This can in fact be proved based on a bootstrapping argument which takes advantage of the particular structure of Eq. (4.1). More details are given in [15].

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