Math1220 Extra Final Review

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1. Evaluate these integrals.

(a) $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^2}$

(b) $\int_0^2 \frac{1}{(x-1)^2}\,dx$

(c) $\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$

(d) $\int x \sin x\, dx$

2. Find the area of the region enclosed by $r = 1 + \sin\theta$.

3. Given this limit $\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$, answer the following questions.

(a) Is this a case where you can use L'Hopital's rule? Why or why not?
(b) Find the limit.
(c) Write out the Maclaurin series for $f(x) = e^x$, term by term.
(d) Write down what $e^x - 1 - x - \frac{x^2}{2}$ is, using your answer for (c) in place of e^x .
(e) Find the limit again using this information from (d).

4. What is the sum of $1 + \frac{\pi}{4} + \frac{\pi^2}{16} + \frac{\pi^3}{64} + \ldots$?

5. Find the area of the region outside $r = 2 + 2\cos\theta$ and inside $r = 2$.

6. (a) Find the first four non-zero terms of $f(x) = e^{x^4}$ in its Maclaurin series.

(b) Use that answer (from (a)) to estimate $\int_0^{\frac{1}{2}} e^{x^4}\,dx$.

7. Evaluate these integrals.

(a) $\int \frac{x^2}{x^3+1}\,dx$

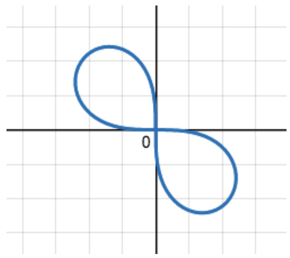
(b) $\int y e^{-2y}\,dy$

(c) $\int_0^{\pi} (\sin\theta + \cos\theta)\, d\theta$

(d) $\int x^2 e^{x^3}\, dx$

(e) $\int \frac{x+1}{x(x-1)(x^2+1)}\, dx$

8. Find the area inside the lemniscate that is given by $r^2 = -9\sin(2\theta)$.



9. Find the convergence set for the following power series.

(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n!}$

(b) $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n+1}$

(c) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{(-3)^n(n^3+1)}$

10. Does $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 3^{n+1}}{e^{2n}}$ converge absolutely, conditionally or diverge?

11. Find a power series for $F(x) = \int x e^{x^3}\, dx$ and state its radius of convergence.

12. Find $\frac{dy}{dx}$ for each function. (Do not bother simplifying your answers! And make sure your answers are given as y = some function of x.)

(a) $y = \frac{\ln(3x+1)}{\cos(\sqrt{x})}$

(b) $y = (x^2+2)^{1+x}$

(c) $y = (\sin(x)+2)^{3x+5}$

(d) $y = 5^{\sinh x} + \frac{1}{1+x^2} + (\arctan(x))^3 + e^{\pi}$

13. Evaluate these integrals.

(a) $\int \frac{\sqrt{4x^2+9}}{x^4}\,dx$

(b) $\int_2^4 \frac{dx}{\sqrt{x-2}}$

(c) $\int_0^{\pi/3} x\cos(3x)\,dx$

(d) $\int \frac{2x+3}{x^3+x}\,dx$

(e) $\int \cos^3 x\,dx$

(f) $\int_{-\infty}^{0} 8x^2 e^{-x^3}\,dx$

(g) $\int_0^1 \frac{4y}{\sqrt{y^2+6}}\,dy$

(h) $\int e^x \sin(e^x)\,dx$

(i) $\int \frac{x+9}{x^3+9x}\,dx$

(j) $\int \frac{\cos x(\sin x+\cos x)}{\sin x}\,dx$

(k) $\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx$

(l) $\int_3^7 \frac{2x}{\sqrt{x-3}}\,dx$

(m) $\int x^2 \ln x\,dx$

(n) $\int_{\frac{1}{2}}^{2} \frac{dx}{x\sqrt[3]{\ln x}}$

14. Find the limit, if it exists.

(a) $\lim_{x\to 0} \frac{\sin(5x)}{\tan(4x)}$

(b) $\lim_{x\to\infty} (2x)^{\frac{1}{3x}}$

(c) $\lim_{x\to 0} (1+\sin x)^{\frac{2}{x}}$

(d) $\lim_{x\to 0} \frac{\sin x-\tan x}{x^2\sin x}$

15. For the sequence given by $a_n = \frac{5n}{\sqrt{5n^2 - 3}}$, write out the first three terms of the sequence. And, determine if the sequence converges or diverges. If it converges, find its limit.

16. Determine if each series is absolutely convergent, conditionally convergent or divergent. (Show your work to convince me of your result and state which tests you're using to get to your answer.)

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{3n}}{n^3 + 5}$

(b) $\sum_{n=1}^{\infty} \frac{n-2}{4n+1}$

(c) $\sum_{n=1}^{\infty} \frac{n^4(-3)^n}{(n+2)!}$

(d) $\sum_{n=1}^{\infty} \frac{(-3)^n n^2}{(2n)!}$

(e) $\sum_{n=1}^{\infty} \frac{2n+7}{\sqrt[4]{4n^4 + 5n + 1}}$

(f) $\sum_{n=1}^{\infty} \frac{3n^3 + 2n}{1 + n^3}$

17. Find a power series that represents $f(x) = \frac{1}{1-2x} + e^{-3x}$. (Write out the terms through x^3.) State its radius of convergence.

18. For $f(x) = \frac{1}{3+x}$

(a) Find the Taylor polynomial of order 3 centered about a = 2.
(b) Approximate f(2.3) using the Taylor polynomial in part (a).
(c) Find a bound for the error in your approximation.

19. For Cartesian coordinates $(-3\sqrt{3}\ ,\ 3)$, find three different ways to represent this point in polar coordinates.

20. Find the slope of the tangent line to the graph $r = 2 + 3\sin\theta$ at the point where $\theta = \frac{\pi}{4}$.

21. For the rectangular (Cartesian) equation $x^2 + (y-2)^2 = 4$

(a) Find the equivalent polar equation for this curve.
(b) **Using the polar equation in part (a)**, find the area of the region bounded by the curve. (Hint: You may want to graph the function to ensure you get the correct integration bounds.)

22. Find the derivative. $D_x\left(x^{1+x}\right)$

23. For $f(x)=x^5+2x^3+4x$, determine if it's an invertible function. If it is invertible, find $(f^{-1})'(7)$.

24. Find the equation of the tangent line to $f(x)=(1+\sin x)^{\cos x}$ at $x=\frac{\pi}{2}$.

25. For the sequence given by $a_n=\frac{1}{\sqrt[3]{n}}+\frac{1}{\sqrt[n]{3}}$, write out the first three terms of the sequence. And, determine if the sequence converges or diverges. If it converges, find its limit.

26. For $f(x)=\frac{2}{x-1}$

(a) Find the Taylor polynomial of order 4 centered about a = 2.
(b) Approximate f(1.5) using the Taylor polynomial in part (a).
(c) Find a bound for the error in your approximation.

27. Convert the point $\left(-1\ ,\ \frac{5\pi}{4}\right)$ from polar coordinates to rectangular coordinates.

28. For the function (in polar coordinates) given by
$r^2-6r\cos\theta-4r\sin\theta+9=0$
(a) convert this function to rectangular coordinates, and
(b) describe the shape.

29. Find a power series that represents $f(x)=\frac{3x^2}{4-x^3}$ and state its radius of convergence.

30. The third degree Taylor polynomial of $f(x)=\ln x$ centered at a = 1 is given by

$$f(x)\approx P_3(x)=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3 \quad .$$

Given that $f^{(4)}(x)=-6x^{-4}$, how closely does this polynomial approximate *f(x)* when x = 1.1? That is, if $R_3(x)=f(x)-P_3(x)$, how large can $|R_3(1.1)|$ be? (In other words, find the error bound on $|R_3(1.1)|$.)

Answers:

1. (a) $\frac{-2}{1+\sqrt{x}}+C$

(b) diverges

(c) 2

(d) $\sin x - x\cos x + C$

2. $\frac{3\pi}{2}$

3. (a) yes, because it's the 0/0 case.

(b) 1/6

(c) $1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+...$

(d) $e^x-1-x-\frac{x^2}{2}=(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+...)-1-x-\frac{x^2}{2}=\frac{x^3}{6}+\frac{x^4}{24}+...$

(e) 1/6

4. $\frac{4}{4-\pi}$

5. $8-\pi$

6. (a) $1+x^4+\frac{1}{2}x^8+\frac{1}{6}x^{12}+\frac{1}{24}x^{16}+...$

(b) ~0.50636007

7. (a) $\frac{1}{3}\ln|x^3+1|+C$

(b) $\frac{-1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}+C$

(c) 2

(d) $\frac{1}{3}e^{x^3}+C$

(e) $\ln\left|\frac{x-1}{x}\right|-\arctan x+C$

8. 9

9. (a) $x\in\mathbb{R}$

(b) (1, 5)

(c) [-2, 4]

10. converges absolutely

11. $\sum_{n=0}^{\infty}\frac{x^{3n+2}}{n!(3n+2)}$

12. (a) $\dfrac{dy}{dx}=\dfrac{\cos(\sqrt{x})\left(\dfrac{3}{3x+1}\right)+\dfrac{\ln(3x+1)\sin(\sqrt{x})}{2\sqrt{x}}}{\cos^2(\sqrt{x})}$

(b) $\dfrac{dy}{dx}=(x^2+x)^{1+x}\left(\dfrac{2x(1+x)}{x^2+2}+\ln(x^2+x)\right)$

(c) $\dfrac{dy}{dx}=(\sin x+2)^{3x+5}\left(3\ln(\sin x+2)+\dfrac{(\cos x)(3x+5)}{\sin x+2}\right)$

(d) $\dfrac{dy}{dx}=5^{\sinh x}(\ln 5)(\cosh x)-\dfrac{2x}{(1+x^2)^2}+\dfrac{3(\arctan x)^2}{1+x^2}$

13. (a) $\dfrac{-(4x^2+9)^{\frac{3}{2}}}{27x^3}+C$

(b) $2\sqrt{2}$

(c) $\dfrac{-2}{9}$

(d) $3\ln|x|-\dfrac{3}{2}\ln(x^2+1)+2\arctan x+C$

(e) $\sin x-\dfrac{1}{3}\sin^3 x+C$

(f) diverges

(g) $4(\sqrt{7}-\sqrt{6})$

(h) $-\cos(e^x)+C$

(i) $\ln|x|-\dfrac{1}{2}\ln(x^2+9)+\dfrac{1}{3}\arctan\left(\dfrac{x}{3}\right)+C$

(j) $\sin(x)+\cos(x)+\ln|csc(x)-\cot(x)|+C$

(k) 2

(l) $\dfrac{104}{3}$

(m) $\dfrac{1}{3}x^3(\ln x-\dfrac{1}{3})+C$

(n) 0

14. (a) $\dfrac{5}{4}$

(b) 1

(c) e^2

(d) $-\dfrac{1}{2}$

15. $a_1=\dfrac{5}{\sqrt{2}}$, $a_2=\dfrac{10}{\sqrt{17}}$, $a_3=\dfrac{15}{\sqrt{42}}$, converges to $\sqrt{5}$

16. (a) converges absolutely

(b) diverges
(c) converges absolutely
(d) converges absolutely
(e) diverges
(f) diverges

17. $2-x+\frac{17}{2}x^2+\frac{7}{2}x^3+\frac{155}{8}x^4+\ldots$, radius of convergence is $\frac{1}{2}$

18. (a) $\frac{1}{5}-\frac{1}{25}(x-2)+\frac{1}{125}(x-2)^2-\frac{1}{625}(x-2)^3+\ldots$
(b) 0.1886768
(c) $|R_4(2.3)|\leq 0.000002592$

19. $(6,\frac{5\pi}{6})$, $(-6,\frac{-\pi}{6})$, $(6,\frac{-7\pi}{6})$

20. $\frac{-2-3\sqrt{2}}{2}$

21. (a) $r=4\sin\theta$
(b) 4π

22. $x^{1+x}\left(\frac{1+x}{x}+\ln x\right)$

23. $\frac{1}{15}$

24. $y=(-\ln 2)x+1+\frac{\pi}{2}\ln 2$

25. $a_1=\frac{4}{3}$, $a_2=\frac{3\sqrt[3]{4}+2\sqrt{3}}{6}$, $a_3=\frac{2}{\sqrt[3]{3}}$ $\left(\text{or } \frac{2\sqrt[3]{9}}{3}\right)$; converges to 1

26. (a) $f(x)\approx 2-2(x-2)+2(x-2)^2-2(x-2)^3+2(x-2)^4$
(b) $\frac{31}{8}$
(c) 4

27. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

28. (a) $(x-3)^2+(y-2)^2=4$
(b) it's a circle with radius of 2 and center at (3, 2)

29. $\sum_{n=0}^{\infty}\frac{3x^{3n+2}}{4^{n+1}}$; radius of convergence = $\sqrt[3]{4}$

30. $|R_3(1.1)|\leq\frac{1}{40000}$ (or 0.000025)