14.4 Green's Theorem

Ex 1 Given $\oint_C \sqrt{y}\,dx + \sqrt{x}\,dy$ where C is the closed curve formed by $y=0$, $x=2$, $y=\frac{x^2}{2}$.

(a) Draw C.

(b) Calculate integral using Green's Thm.

Thm let
- R be region in xy-plane
- C is simple, closed curve enclosing R (w/ parameterization $\vec{r}(t)$)
- $\vec{F}(x,y) = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$ be continuously differentiable over $R \cup C$.

Form ① (flux across boundary)

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \nabla\cdot\vec{F}\,dA$$

$\vec{n}$ = unit normal vector to C

$$\Longleftrightarrow \oint M\,dy - N\,dx = \iint_R (M_x + N_y)\,dx\,dy$$

Form ② (circulation along boundary)

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \nabla\times\vec{F}\cdot\hat{k}\,dA$$

$$\Longleftrightarrow \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dx\,dy$$

Big idea = we can exchange line integral for double integral

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area of R:

$$A = \oint \left(-\tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy\right)$$

14.4 (cont)

Ex 2 Given vector field

$\vec{F}(x,y) = x\hat{\imath} + 2y\hat{\jmath}$ and curve C given by $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $t \in [0, 2\pi]$

(a) Draw vector field, curve C and make predictions about flux and circulation.

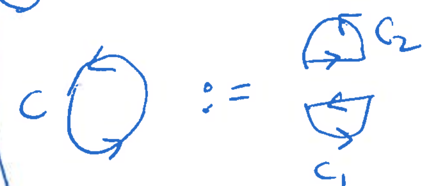
(b) Calculate $\oint_C \vec{F} \cdot \hat{n}\, ds$ (flux across boundary)

Idea of proof of Green's Thm:

① Observation:

$$\int_{(a,b)}^{(c,d)} f\, ds = -\int_{(c,d)}^{(a,b)} f\, ds$$

② Subdivide regions

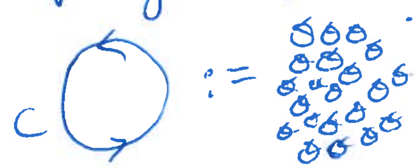


$C = C_1 \cup C_2$

$$\Rightarrow \oint_C f\, ds = \oint_{C_1} f\, ds + \oint_{C_2} f\, ds$$

why? ______

③ Subdivide into infinitely many subregions.



$\Rightarrow$ line integral becomes double integral over a closed 2-d region

14.4 (cont)

Ex 2 (c) Calculate $\oint_C \vec{F} \cdot \vec{T}\, ds$ (circulation along boundary)

Ex 3 Find area (using Green's Thm) between $y = \sqrt{x}$ and $y = \frac{1}{4}x$. Check your answer using a different method.

14.5 Surface Integrals

Ex1 Evaluate

$\iint_G g(x,y,z)\,dS$ given $g(x,y,z)=y$,

$G:\ z=4-y^2,\ 0\le x\le 3,\ 0\le y\le 2$

Thm

Let

- R be closed, bndd region in xy-plane
- f be a fn with first-order partial derivatives on R
- G be a surface over R given by $z=f(x,y)$
- $g(x,y,z)=g(x,y,f(x,y))$ is continuous on R

Then

$$\iint_G g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\underbrace{\sqrt{f_x^2+f_y^2+1}\,dy\,dx}_{dS}.$$

14.5 (cont)

Ex 2 Evaluate $\iint_G 3z\, dS$ where G is the top of the tetrahedron bounded by coordinate planes and plane $2x + 6y + 3z = 6$.

Ex3 Evaluate flux across G where

$F(x,y,z) = 2\hat{i} + 5\hat{j} + 3\hat{k}$, G is the part of the cone

$z = \sqrt{x^2+y^2}$ outside the cylinder $x^2+y^2=1$ and

inside the cylinder $x^2+y^2=4$.

flux of $\vec{F}$ across G

$$\iint_G \vec{F}\cdot\hat{n}\,dS$$

$$= \iint_R [-Mf_x - Nf_y + P]\,dx\,dy$$

where

- $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$
- G is surface $f(x,y)=z$.
- $\vec{n}$ is upward unit normal on G.
- $f(x,y)$ has continuous 1st order partial derivatives

14.6 Gauss' Divergence Theorem

Ex 1 Let $\vec{F}(x,y,z) = 4z\hat{k}$ and S be upper hemisphere with radius 3 and center $(0,0,0)$.

(a) Calculate

$$\iint_{\partial S} \vec{F} \cdot \hat{n} \, dS$$

(as a surface integral!!)

Thm

Let

- $\vec{F}(x,y,z)$ be vector field continuously differentiable in solid S
- S a 3-d solid
- ∂S boundary of S (a surface)
- $\hat{n}$ unit outer normal to ∂S

Then

$$\iint_{\partial S} \vec{F}(x,y,z) \cdot \hat{n} \, dS = \iiint_S \operatorname{div} \vec{F} \, dV$$

$(dV = dx\,dy\,dz)$

(3-d analog of Green's Thm)

14.6 (cont)

Ex1 (cont)

(b) Calculate $\iint_{\partial S} \vec{F}\cdot\hat{n}\,dS$ using Gauss' Thm.

Ex2 Calculate $\iint_{\partial S} \vec{F}\cdot\hat{n}\,dS$ where $\vec{F} = xy\,\hat{i} + e^{x}\hat{j} + z^{3}\hat{k}$

over the box $\{(x,y,z) \mid x\in[0,3], y\in[1,2], z\in[0,1]\}$

14.6 (cont)

Ex3 Calculate $\iint_{\partial S} \vec{F}\cdot\hat{n}\,dS$ where $\vec{F} = 3x\hat{i} + z\hat{j} + 2z^2\hat{k}$ and S is solid between paraboloid $z = 4 - x^2 - y^2$ and cylinder $x^2 + y^2 = 1$ and xy-plane.

14.7 Stokes Theorem

Ex1 Use Stokes' Thm to calculate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$ where $\vec{F} = \langle xy, yz, xz \rangle$, S is the triangular surface (part of a plane) with vertices $(0,0,0)$, $(1,0,0)$ and $(0,2,1)$, $\hat{n}$ is upper normal.

Let

- S be 3-d surface
- $\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$
- M, N, P have continuous 1st order partial derivatives
- C is piece-wise smooth simple, closed curve, positively oriented.
- $\hat{T}$ is unit tangent vector to C.

Then

$$\oint_C \vec{F} \cdot \hat{T}\, ds = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

$$= \iint_R (\nabla \times \vec{F}) \cdot \vec{n}\, dx\, dy$$

Remember

$$\oint_C \vec{F} \cdot \hat{T}\, ds$$

$$= \int_C (M\,dx + N\,dy + P\,dz)$$

14.7 (cont)

Ex2 Use Stokes' Thm to calculate $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$

where $\vec{F} = \langle z-y,\ z+x,\ -(x+y)\rangle$

S is part of paraboloid $z=2-x^2-y^2$ above

$z=1$ plane, and $\hat{n}$ is upward normal.

Ex 3 Use Stokes' Thm to calculate $\oint_C \vec{F} \cdot \hat{T}\, ds$

where $\vec{F} = (x^2+y^2)\hat{\imath} - x(x^2+y^2)\hat{\jmath} + 0\hat{k}$

C is the rectangular path from $(0,0,0)$ to $(1,0,0)$ to $(1,1,1)$ to $(0,1,1)$ to $(0,0,0)$.