14.1 Vector Fields

Ex 1 Fill in the table.

Let $f(x,y,z)$ be a scalar field and
$\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$
be a vector field.

	Notation	formula	input	output
gradient of f				
divergence of $\vec{F}$				
curl of $\vec{F}$				

14.1 (cont)

Ex2 Sketch sample of vectors for the given vector field $\vec{F}$.

(a) $\vec{F}(x,y) = x\hat{\imath} - y\hat{\jmath}$

(b) $\vec{F}(x,y) = -2\hat{\jmath}$

(c) $\vec{F}(x,y,z) = 2\hat{\jmath} + z\hat{k}$ (try to draw vectors w/ starting pts in xy-, yz-, and xz-planes.

14.1 (cont)

Ex 3 let $\vec{F}(x,y,z) = xyz\,\hat{i} + 2y^2\,\hat{j} - 3x^2 z\,\hat{k}$

find:

(a) $\operatorname{div} \vec{F}$

(b) $\operatorname{curl} \vec{F}$

(c) $\operatorname{grad}(\operatorname{div} \vec{F})$

(d) $\operatorname{div}(\operatorname{curl} \vec{F})$

14.2 Line Integrals

Ex1 Evaluate

$\int_C x e^y \, ds$ where C is line segment from $(-1,2)$ to $(1,1)$.

C given by $x = x(t)$, $y = y(t)$, $t \in [a,b]$

$$\int_C f(x,y)\,ds$$

$$= \int_a^b f(x(t), y(t))\,ds$$

where

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

What does this line integral measure?

Let $\vec{F} = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$ (force)

$M = M(x,y,z)$, $N = N(x,y,z)$

$P = P(x,y,z)$

$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$

work

$$W = \int_C \vec{F} \cdot d\vec{r}$$

(work done moving a particle over curve C w/ force $\vec{F}$)

14.2 (cont)

Ex 2 Evaluate $\int_C [xz\,dx + (y+z)\,dy + x\,dz]$ where C is the curve $x = e^t$, $y = e^{-t}$, $z = e^{2t}$, $t \in [0,1]$

Ex 3 Find the work done by force field $\vec{F}(x,y,z) = (2x-y)\hat{\imath} + 2z\hat{\jmath} + (y-z)\hat{k}$ when moving a particle along the line segment from $(0,0,0)$ to $(1,4,5)$.

14.3 Independence of Path

Fundamental Thm of Line Integrals

C is curve given by $\vec{r}(t)$, $t \in [a,b]$; $\vec{r}\,'(t)$ exists.
If $f(\vec{r})$ is continuously differentiable on an open set containing C, then
$$\int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(\vec{b}) - f(\vec{a}).$$

Equivalent Conditions

$\vec{F}(\vec{r})$ continuous on open connected set D. Then

(a) $\vec{F} = \nabla f$ for some fn f.
(i.e. $\vec{F}$ is conservative)

$\Leftrightarrow$ (b) $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is indep. of path in D

$\Leftrightarrow$ (c) $\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = 0$ $\forall$ closed paths in D.

Thm $\vec{F} = M\hat{\imath} + N\hat{\jmath} + P\hat{k}$ continuously differentiable on open, simply connected set D. $\vec{F}$ conservative $\Leftrightarrow \nabla \times \vec{F} = \vec{0}$

(in 2-d, $\nabla \times \vec{F} = \vec{0}$ iff $M_y = N_x$)

Qns

① What does it mean to be indep. of path?

② Why does D need to be open & simply connected?

③ If $\vec{F}$ is conservative, what is it conserving?

④ Why didn't the Thm (left) get grouped w/ equivalent conditions?

14.3 (cont)

Ex1 Determine if the given field is conservative. If so, find f s.t. $\vec{F} = \nabla f$.

(a) $\vec{F}(x,y) = \left(x + \frac{1}{(x+y)^2}\right)\hat{i} + \left(3 + \frac{1}{(x+y)^2}\right)\hat{j}$

(b) $\vec{F}(x,y) = 4y^2\cos(xy^2)\hat{i} + 8x\cos(xy^2)\hat{j}$

14.3 (cont)

Ex 2 Use $\vec{F}(x,y) = \left(x + \frac{1}{(x+y)^2}\right)\hat{i} + \left(3 + \frac{1}{(x+y)^2}\right)\hat{j}$

(a) What is largest open, connected set on which $\vec{F}(x,y)$ is continuous?

(b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ using Fundamental Thm of Line Integrals. (Why are we sure we can use FTLI?) C is the curve $\vec{r} = t^2\hat{i} + 2t^3\hat{j}$, $t \in [1,2]$.

(c) How would you calculate $\int_C \vec{F} \cdot d\vec{r}$ w/o FTLI?

Ex 3 Show that the line integral is independent of path.

$$\int_{(0,0,0)}^{(\pi,\pi,0)} \left[(\cos x + 2yz)\,dx + (\sin y + 2xz)\,dy + (z + 2xy)\,dz \right].$$

Then evaluate it.