Since $0 < r < 1$, $\Sigma_{n=N}^{\infty} r^n$ is a convergent geometric series, and hence by the Comparison Test (12.22), $\Sigma_{n=N}^{\infty} |a_n|$ converges. Consequently $\Sigma_{n=1}^{\infty} |a_n|$ converges; that is, $\Sigma\, a_n$ is absolutely convergent. This proves (i). The remainder of the proof is left as an exercise.

Example 6 Determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$.

Solution Since all terms are positive we may delete the absolute value signs in the Root Test (12.30). Thus

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^{3n+1}}{n^n}} = \lim_{n\to\infty} \left(\frac{2^{3n+1}}{n^n}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{2^{3+(1/n)}}{n} = 0 < 1.$$

Hence the series converges by the Root Test.

As a final remark, it can be proved that if a series $\Sigma\, a_n$ is absolutely convergent and if the terms are rearranged in any manner, then the resulting series converges and has the same sum as the given series. This is not true for conditionally convergent series. Indeed, if $\Sigma\, a_n$ is conditionally convergent, then by suitably rearranging terms one can obtain either a divergent series, or a series which converges and has any desired sum S.*

EXERCISES 12.5

In Exercises 1–38, determine whether (a) a series which contains both positive and negative terms is absolutely convergent, conditionally convergent, or divergent; or (b) a positive term series is convergent or divergent.

1 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

2 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n^2+1}$

3 $\displaystyle\sum_{n=2}^{\infty} (-1)^{n} \frac{5}{n^4-1}$

4 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} e^{-n}$

5 $\displaystyle\sum_{n=1}^{\infty} \frac{1000-n}{n!}$

6 $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$

7 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3n+1}{2^n}$

8 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n^2+4}$

9 $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n(3^{n+1})}$

10 $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{5^n(n+1)}$

* See, for example, R. C. Buck, *Advanced Calculus*, Third Edition (New York: McGraw-Hill, 1978), pp. 238–239.

(b) Since the series contains only positive terms the absolute value signs in Theorem (12.29) may be deleted. Thus

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^2} \cdot \frac{n^2}{3^n}$$

$$= \lim_{n\to\infty} \frac{3n^2}{n^2+2n+1} = 3 > 1$$

and hence the series diverges by the Ratio Test.

Example 5 Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution Since all terms of the series are positive, we may drop the absolute value signs in the Ratio Test (12.29). Thus

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)} \cdot \frac{1}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

where the last equality is a consequence of Theorem (8.32). Since $e > 1$, the given series diverges by the Ratio Test.

The following test is very useful if a_n contains only powers of n. It is not as versatile as the Ratio Test because it cannot be applied if a_n contains factorials.

(12.30) The Root Test

Let $\Sigma\, a_n$ be an infinite series.

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, the series is absolutely convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, the series is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the series may be absolutely convergent, conditionally convergent, or divergent.

Proof. The proof is similar to that used for the Ratio Test. If $L < 1$ as in (i), let us consider any number r such that $L < r < 1$. By the definition of limit, there exists a positive integer N such that if $n \geq N$,

$$\sqrt[n]{|a_n|} < r, \quad \text{or} \quad |a_n| < r^n.$$