Key Definitions: Sections 3.1-3.3, 4.1-4.7, 5.1-5.2

- The determinant of an $n \times n$ matrix is

- The (i, j)-cofactor of an $n \times n$ matrix A is f a vector space is

- A vector space is

- A subspace of a vector space is

- The null space of an $m \times n$ matrix is

- The column space of an $m \times n$ matrix is

- A linear transformation is

- A basis is

- The $\mathcal{B}-$coordinates of $\mathbf{x}$ are

- The dimension of a vector space V is

- The rank of A is

- An eigenvector of A is

- An eigenvalue of A is

- Two matrices are similar if

Major Theorems: Sections 3.1-3.3, 4.1-4.7, 5.1-5.2

Chapter 3

Theorem 1 Cofactor Expansion *The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.*

Cofactor expansion across the i^{th} **row** *is given by:*

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

Cofactor expansion across the j^{th} **column** *is given by:*

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Theorem 2 *If A is a triangular matrix, then det(A) is the product of the entries on the main diagonal of A.*

Theorem 3 ***Determinant Properties*** *Let A and C be $n \times n$ matrices.*

(a) If a multiple of one row of A is added to another row to produce a matrix B, then

$$det(B) = \underline{\hspace{6cm}}$$

(b) If two rows of A are interchanged to produce B, then

$$det(B) = \underline{\hspace{6cm}}$$

(c) If one row of A is multiplied by k to produce B, then

$$det(B) = \underline{\hspace{6cm}}$$

(d) The determinant of the transpose of A is,

$$det(A^T) = \underline{\hspace{6cm}}$$

(e) The determinant of the product AC is,

$$det(AC) = \underline{\hspace{6cm}}$$

Theorem 4 IMT extended *A square matrix A is invertible if and only if $det(A) \neq 0$.*

Theorem 5 ***Cramer's Rule*** *Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by*

$$x_i = \frac{det(A_i(\mathbf{b}))}{det(A)}, \quad i = 1, 2, \ldots, n.$$

$A_i(\mathbf{b})$ is defined as the matrix where the ith column of A is replaced by $\mathbf{b}$. That is,

$$A_i(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{b} & \ldots & \mathbf{a}_n \end{bmatrix}.$$

Theorem 6 An Inverse Formula *Let A be an invertible $n \times n$ matrix. Then,*

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

where $adj(A)$ denotes the adjugate (or classical adjoint), the $n \times n$ matrix of cofactors $C^T = [C_{ji}]$.

Theorem 7 Area or Volume

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|det(A)|$.

If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|det(A)|$.

Theorem 8 Expansion Factors

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation determined by the 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{area\ of\ T(S)\} = |det(A)| \cdot \{area\ of\ S\}$$

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation determined by the 3×3 matrix A. If S is a parallelepiped in $\mathbb{R}^3$, then

$$\{volume\ of\ T(S)\} = |det(A)| \cdot \{volume\ of\ S\}$$

Chapter 4

Theorem 1 *If V is a vector space, and $\mathbf{v_1}, \dots, \mathbf{v_p} \in V$, then span$\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ is a subspace of V.*

Note: we call span$\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ the subspace spanned by $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$.

Theorem 2 *The null space of an $m \times n$ matrix is a subspace of* _____

Theorem 3 *The column space of an $m \times n$ matrix is a subspace of* _____

Theorem 4 *An indexed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors with $\mathbf{v}_1 \neq \mathbf{0}$ is linearly dependent if and only if some vector $\mathbf{v}_j$ with $j > 1$ is a linear combination of the preceding $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.*

Theorem 5 ***Spanning Set Theorem*** *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\} \subset V$ and $H =$ span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.*

- *If some $\mathbf{v}_k \in S$ is a linear combination of the remaining vectors in S, the set formed by removing $\mathbf{v}_k$ still spans H.*
- *If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.*

Theorem 6 *The pivot columns of a matrix A form a basis for Col A.*

Theorem 7 ***The Unique Representation Theorem***

*Let $\mathcal{B} = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$ be a basis for a vector space V. Then, for each $\mathbf{x} \in V$, there exist **unique** $c_1, \dots, c_n \in \mathbb{R}$ such that*

$$\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}.$$

Theorem 8 *Let* $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ *be a basis for a vector space* V*. The coordinate mapping* $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ *is a* ***one-to-one*** *linear transformation from* V ***onto*** $\mathbb{R}^n$*.*

Theorem 9 *If a vector space* V *has a basis* $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$*, then any set in* V *containing more*

than n *vectors must be* ________________________________

Theorem 10 *If a vector space* V *has a basis of* n *vectors, then every basis of* V *must consist of*

exactly _____ *vectors.*

Theorem 11 *Let* H *be a subspace of a finite-dimensional vector space* V*. Any linearly independent set in* H *can be expanded to a basis for* H*. Also,* H *is finite-dimensional and* $\dim H \leq \dim V$*.*

Theorem 12 ***The Basis Theorem***
Let V *be a* p*-dimensional vector space where* $p \geq 1$*. Any linearly independent set of exactly* p *elements in* V *is automatically a basis for* V*. Any set of exactly* p *elements that spans* V *is automatically a basis for* V*.*

Theorem 13 *If two matrices* A *and* B *are row equivalent, then their row spaces are the same. If* B *is in echelon form, the nonzero rows of* B *form a basis for the row space of* A *as well as for that of* B*.*

Theorem 14 ***The Rank-Nullity Theorem*** *Let* A *be an* $m \times n$ *matrix.*

$$rank\ A + dim(Nul\ A) = \_\_\_\_\_$$

Theorem 15 ***Change of Basis*** *Let* $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ *and* $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ *be bases of a vector space* V*. Then, there is a unique* $n \times n$ *matrix* $P_{\mathcal{C} \longleftarrow \mathcal{B}}$ *such that*

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C} \longleftarrow \mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

where the columns $P_{\mathcal{C} \longleftarrow \mathcal{B}}$ *are the* $\mathcal{C}$*-coordinate vectors of the vectors in the basis* $\mathcal{B}$*. That is,*

$$P_{\mathcal{C} \longleftarrow \mathcal{B}} = [[\mathbf{b}_1]_{\mathcal{C}} \ [\mathbf{b}_2]_{\mathcal{C}} \ldots [\mathbf{b}_n]_{\mathcal{C}}]$$

$P_{\mathcal{C} \longleftarrow \mathcal{B}}$ *is called the* <u>*change-of-coordinates matrix from* $\mathcal{B}$ *to* $\mathcal{C}$</u> *and is invertible.*

Chapter 5

Theorem 1 *The eigenvalues of a triangular matrix are the entries on the main diagonal.*

Theorem 2 *If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is linearly independent.*

Theorem 3 *If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial*

and hence, the same ____________________ (with the same multiplicities).

The Invertible Matrix Theorem (continued)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ ____________________ matrix.

(c) A has ____________________ postions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the ____________________ solution.

(e) The columns of A form a linearly ____________________ set.

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is ____________________.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has ____________________ solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A ____________________ $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ ____________________ $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA =$ ____________________.

(k) There is an $n \times n$ matrix D such that $AD =$ ____________________.

(l) A^T is an ____________________ matrix.

(m) The ____________________ of A form a basis of __________.

(n) Col $A =$ __________

(o) $\dim(\text{Col } A) =$ __________

(p) rank $A =$ __________

(q) Nul $A =$ __________

(r) $\dim(\text{Nul } A) =$ __________

(s) __________ is not an eigenvalue of A.

(t) $det(A) \neq$ __________

Supplemental Practice Problems:

1. Let A and B be 5×5 matrices with $\det A = -2$ and $\det B = 3$. Use properties of determinants to compute each of the following:
   (a) $\det BA$
   (b) $\det 2A$

2. Suppose that all of the entries in an invertible matrix A are integers and $\det(A) = 1$. Using the adjugate formula for A^{-1}, $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$, explain why all the entries of A^{-1} are integers.

3. Compute the area of the parallelogram whose vertices are given by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

4. Use Cramer's Rule to solve the matrix equation

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & -1 \\ 0 & 3 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ -6 \\ 5 \end{bmatrix}$$

5. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ using the adjugate formula, $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$.

6. Let $M_{2\times2}$ be the space of 2×2 matrices with real entries. $M_{2\times2}$ has natural operations of matrix addition and scalar multiplication, and with these operations, $M_{2\times2}$ is a vector space. (You do not need to check this.) Consider the subset of symmetric matrices:

$$H = \{A \in M_{2\times2} : A^T = A\}.$$

   Is H a subspace of $M_{2\times2}$? Justify your answer.

7. Let W be the set of all vectors of the form given below. Find a set S of vectors that spans W or give an example to show that W is not a vector space.
   (a) $W = \left\{ \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix} : a, b \in R \right\}$
   (b) $W = \left\{ \begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix} : a, b, c \in R \right\}$

8. Constructions:
   (a) Give an example of a basis $\{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}\}$ of $\mathbb{P}_2$ such that $[t^2]_\mathcal{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.
   (b) Give an example of a vector space V whose objects are matrices such that $\dim V = 100$.
   (c) Give an example of a 2×2 matrix A such that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector of A with associated eigenvalue $\lambda = 4$.

(d) Give an example of a matrix A that has a two-dimensional eigenspace.

(e) Give an example of a 2×2 matrix A with real eigenvalues that is not diagonalizable.

9. Let $M_{2\times2}$ be the space of 2×2 matrices with real entries. $M_{2\times2}$ has natural operations of matrix addition and scalar multiplication, and with these operations, $M_{2\times2}$ is a vector space. (You do not need to check this.) Consider the subset of symmetric matrices:

$$H = \{A \in M_{2\times2} : A^T = A\}.$$

Is H a subspace of $M_{2\times2}$? Justify your answer.

10. For each vector space, determine if the given set is a basis. Justify your answer.

(a) $\mathbb{R}^2 : \left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -9 \\ 6 \end{bmatrix} \right\}$

(b) $\mathbb{R}^2 : \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

(c) $\mathbb{R}^2 : \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$

(d) $\mathbb{R}^3 : \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}$

(e) $\mathbb{R}^3 : \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$

11. For each space, find a basis and express the redundant vectors as linear combinations of the basis vectors.

(a) $\text{span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

(b) $\text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}$

12. Consider the matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & 6 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Find the eigenvalues of A and their corresponding eigenspaces.

13. Consider the set $\mathcal{B} = \mathbf{u}, \mathbf{v}$ where $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 6 \end{bmatrix}$.

(a) Explain why $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

(b) Express the vectors $\begin{bmatrix} 3 \\ -4 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in terms of the basis $\mathcal{B}$.

(c) Sketch the grid associated to $\mathcal{B}$ along with the vectors from part(b).

14. For each matrix, compute its rank and find a basis for its column space; compute its nullity and find a basis for its null space.

(a) $\begin{bmatrix} 1 & 2 & -3 & -2 & 1 \\ -1 & -2 & 5 & 6 & 3 \\ 3 & 6 & -2 & 8 & 17 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 6 & -3 \\ 2 & 7 & -1 \\ 3 & 8 & 1 \\ 4 & 9 & 3 \\ 5 & 10 & 5 \end{bmatrix}$

15. Consider the set $\mathcal{B} = \{1+t, 1+t^2, t+t^2\}$ in $\mathbb{P}_2$.
    (a) Is $\mathcal{B}$ a basis for $\mathbb{P}_2$? Show work in support of your answer.
    (b) Compute the $\mathcal{B}$-coordinates of the vector $2+2t+2t^2 \in \mathbb{P}_2$.

16. Consider the matrix A and its factorization

$$A = \begin{bmatrix} -2 & 4 & 0 \\ 0 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

    Compute A^9.

17. Let $A = \begin{bmatrix} 1 & 6 & -3 \\ 0 & 0 & 0 \\ -2 & -12 & 6 \end{bmatrix}$.
    (a) Find the reduced row echelon form of A.
    (b) Find a basis for Nul$A$. What is dim(Nul$A$)?
    (c) Find a basis for Col$A$. What is dim(Col$A$)?
    (d) Find a basis for Row$A$. What is dim(Row$A$)?

18. Is the set of polynomials $\{1+2t, 3t+3t^2, 4t+7t^2\}$ a basis for $\mathbb{P}_2$? Show supporting work!

19. Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ of $\mathbb{R}^2$.
    (a) Compute the $\mathcal{B}$-coordinates of each of the following vectors in $\mathbb{R}^2$: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$
    (b) Write down the matrix $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ that converts $\mathcal{B}$-coordinates to standard coordinates. Multiply $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ by one of your answers in the previous part to check that you get back the original vector.
    (c) Compute the matrix $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P}$ that converts standard coordinates to $\mathcal{B}$-coordinates. Compute $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. You should get your answers in the first part.

20. Let $\mathbb{P}_2 \xrightarrow{T} \mathbb{R}^2$ be the linear transformation defined by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(2) \end{bmatrix}$.

   (a) Compute $T(1), T(t)$, and $T(t^2)$. Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$. Using the coordinate mapping $\mathbb{P}_2 \xrightarrow{[\ ]_{\mathcal{E}}} \mathbb{R}^3$, we can view T as a linear transformation $\mathbb{R}^3 \xrightarrow{S} \mathbb{R}^2$ defined by $S(\mathbf{x}) = A\mathbf{x}$.

   (b) Write down the matrix A. Hint: the columns of A are your answers in (a).

   (c) Use the matrix A to check that $S([t^2]_{\mathcal{E}}) = T(t^2)$.

   (d) Compute a basis for Nul A.

   (e) Use your answer in (d) to write down a basis for the kernel of T.

21. Write down the equation that defines what it means for $\mathbf{x}$ to be an eigenvector of matrix A with associated eigenvalue λ.

22. Show that $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 7 \\ 0 & -1 \end{bmatrix}$ and compute its eigenvalue λ.

23. Suppose $\lambda = 6$ is an eigenvalue for the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 \\ -6 & 6 & 4 \\ -3 & 0 & 8 \end{bmatrix}.$$

   Compute a basis for the associated eigenspace.

24. Let $A = \begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$.

   (a) Compute the eigenvalues λ_1, λ_2 of A.

   (b) Compute the associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of A.

25. Is it possible for a nonhomogeneous system of seven equations in six unknowns to have a unique solution for some right-hand side of constants? Is it possible for such a system to have a unique solution for every right-hand side? Justify your answer.

26. Determine whether each subset of vectors is a subspace of a vector space. If so, find the dimension of the subspace and identify the vector space.

   (a) $\text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}$, (b) Row A, (c) Nul A^T, (d) $\left\{ \begin{bmatrix} 4a+3b \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$,

   (e) $\{y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) : c_1, c_2 \in \mathbb{R}\}$

27. Define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$ where A is a 3×3 matrix with eigenvalues 5 and -2. Does there exist a basis $\mathcal{B}$ for $\mathbb{R}^3$ such that the $\mathcal{B}$-matrix for T is a diagonal matrix? Justify your answer.