Recall that a random variable $X$ is continuous if
1). possible values of $X$ comprise either a single interval on the number line (for some $A < B$, any number $x$ between $A$ and $B$ is a possible value) or a union of disjoint intervals;
2). $P(X = c) = 0$ for any number $c$ that is a possible value of $X$.

Examples:
1. $X =$ the temperature in one day. $X$ can be any value between $L$ and $H$, where $L$ represents the lowest temperature and $H$ represents the highest temperature.
2. Example 4.3: $X =$ the amount of time a randomly selected customer spends waiting for a haircut before his/her haircut commences.
   Is $X$ really a continuous rv? No.
   The point is that there are customers lucky enough to have no wait whatsoever before climbing into the barber’s chair, which means $P(X = 0) > 0$. Only conditioned on no chairs being empty, the waiting time will be continuous.
Let’s consider the temperature example again. We want to know the probability that the temperature is in any given interval. For example, what’s the probability for the temperature between 70°C and 80°C? Ultimately, we want to know the probability distribution for $X$. One way to do that is to record the temperature from time to time and then plot the histogram. However, when you plot the histogram, it’s up to you to choose the bin size. But if we make the bin size finer and finer (meanwhile we need more and more data), the histogram will become a smooth curve which will represent the probability distribution for $X$. 
Probability Density Functions

(a) bin size 2

(b) bin size 0.8

(c) limit case
Definition

Let $X$ be a continuous rv. Then a **probability distribution** or **probability density function (pdf)** of $X$ is a function $f(x)$ such that for any two numbers $a$ and $b$ with $a \leq b$,

$$
P(a \leq X \leq b) = \int_a^b f(x)dx$$

That is, the probability that $X$ takes on a value in the interval $[a, b]$ is the area above this interval and under the graph of the density function. The graph of $f(x)$ is often referred to as the **density curve**.
Figure: \( P(60 \leq X \leq 70) \)
Remark:
For \( f(x) \) to be a legitimate pdf, it must satisfy the following two conditions:
1. \( f(x) \geq 0 \) for all \( x \);
2. \( \int_{-\infty}^{\infty} f(x) \, dx = \text{area under the entire graph of } f(x) = 1. \)
Probability Density Functions

Example:
A clock stops at random at any time during the day. Let \( X \) be the time (hours plus fractions of hours) at which the clock stops. The pdf for \( X \) is known as

\[
f(x) = \begin{cases} 
\frac{1}{24} & 0 \leq x \leq 24 \\
0 & \text{otherwise}
\end{cases}
\]

The density curve for \( X \) is showed below:
Example: (continued)
A clock stops at random at any time during the day. Let $X$ be the time (hours plus fractions of hours) at which the clock stops. The pdf for $X$ is known as

$$f(x) = \begin{cases} \frac{1}{24} & 0 \leq x \leq 24 \\ 0 & \text{otherwise} \end{cases}$$

If we want to know the probability that the clock will stop between 2:00pm and 2:45pm, then

$$P(14 \leq X \leq 14.75) = \int_{14}^{14.75} \frac{1}{24} \, dx = \frac{1}{24} \left|^{14.75}_{14} \right| = \frac{1}{32}$$
Probability Density Functions

Definition

A continuous rv $X$ is said to have a **uniform distribution** on the interval $[A, B]$, if the pdf of $X$ is

$$f(x; A, B) = \begin{cases} \frac{1}{A-B} & A \leq x \leq B \\ 0 & \text{otherwise} \end{cases}$$

The graph of any uniform pdf looks like the graph in the previous example:
Comparisons between continuous rv and discrete rv:
For discrete rv $Y$, each possible value is assigned positive probability;
For continuous rv $X$, the probability for any single possible value is 0!

$$P(X = c) = \int_c^c f(x) \, dx = \lim_{\epsilon \to 0} \int_{c-\epsilon}^{c+\epsilon} f(x) \, dx = 0$$

Since $P(X = c) = 0$ for continuous rv $X$ and $P(Y = c') > 0$, we have

$$P(a \leq X \leq b) = P(a < X < b) = P(a < X \leq b) = P(a \leq X < b)$$

while $P(a' \leq Y \leq b')$, $P(a' < Y < b')$, $P(a' < Y \leq b')$ and $P(a' \leq Y < b')$ are different.