Extreme first-passage times for random walks on networks

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Many biological, social, and communication systems can be modeled by "searchers" moving through a complex network. For example, intracellular cargo is transported on tubular networks, news and rumors spread through online social networks, and the rapid global spread of infectious diseases occurs through passengers traveling on the airport network. To understand the timescale of search (or "transport" or "spread"), one commonly studies the first-passage time (FPT) of a single searcher (or "transporter" or "spreader") to a target. However, in many scenarios the relevant timescale is not the FPT of a single searcher to a target, but rather the FPT of the fastest searcher to a target out of many searchers. For example, many processes in cell biology are triggered by the first molecule to find a target out of many, and the time it takes an infectious disease to reach a particular city depends on the first infected traveler to arrive out of potentially many infected travelers. Such fastest FPTs are called extreme FPTs. In this paper, we study extreme FPTs for a general class of continuous-time random walks on networks (which includes continuous-time Markov chains). In the limit of many searchers, we find explicit formulas for the probability distribution and all the moments of the kth fastest FPT for any fixed $k \geqslant 1$. These rigorous formulas depend only on network parameters along a certain geodesic path(s) from the starting location to the target since the fastest searchers take a direct route to the target. Hence, the extreme FPTs are independent of the details of the network outside this geodesic(s) and can be drastically faster and less variable than conventional FPTs of single searchers. Furthermore, our results allow one to estimate if a particular system is in a regime characterized by fast extreme FPTs. We also prove similar results for mortal searchers on a network that are conditioned to find the target before a fast inactivation time. We illustrate our results with numerical simulations and uncover potential pitfalls of modeling diffusive or subdiffusive processes involving extreme statistics. In particular, we find that the many searcher limit does not commute with the diffusion limit for random walks, and thus care must be taken when choosing spatially continuous versus spatially discrete diffusion models.

I. INTRODUCTION

Networks are used to model many biological, social, and communication systems [1–3]. A network (or graph) consists of (i) nodes (or vertices) which can represent cells, individuals, cities, computers, etc., and (ii) edges between nodes which represent their interactions. An important area of network science studies *random walks on networks*, which are stochastic processes involving "searchers" (or "transporters" or "spreaders") who explore a network by randomly moving between connected nodes [4,5]. Random walks on networks have been used to model very diverse dynamical systems, including the spread of infectious diseases [6], intracellular transport [7], animal foraging, the spread of opinions and rumors, and node ranking (i.e., PageRank) [5].

First passage times (FPTs) are commonly used to understand the timescale of search (or "transport" or "spread") in such models [4,5,8]. A FPT is defined as the first time a searcher reaches a given target(s). Mathematically, if $X(t)$ denotes the position of a searcher in a discrete set of nodes I at time $t \geqslant 0$, then the FPT to some set of target nodes $I_{\text{target}} \subset I$

is defined by

$$\tau := \inf\{t > 0 : X(t) \in I_{\text{target}}\}. \tag{1}$$

Important advances have been made to compute the statistics of such a FPT, and in particular to understand how the FPT depends on the structure of the network [4,5,9,10].

However, in many applications the important timescale is not the FPT of a given single searcher to a target. Instead, the relevant timescale is the FPT of the fastest searcher to find a target out of many searchers. For example, in the context of epidemics spreading globally between cities, the first time the disease reaches a particular locale depends on the first infected traveler to arrive out of potentially many infected travelers. As another example, it was recently pointed out that the transport efficiency of the endoplasmic reticulum network depends on the time it takes the fastest molecule to travel between nodes out of many molecules [7]. In fact, many events in cell biology are triggered by the fastest searcher out of many searchers (see the review [11] and subsequent commentaries [12–18]).

Such fastest FPTs are an example of extreme statistics [19–21] and are thus called extreme FPTs [22]. Mathematically, an extreme FPT is defined as

$$T_N := \min\{\tau_1, \ldots, \tau_N\},$$

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where $\tau_1, \ldots, \tau_N$ are N FPTs as in (1). In particular, $\tau_1, \ldots, \tau_N$ represent the FPTs of N searchers who are searching simultaneously for a target, and thus T_N is the FPT of the fastest searcher. Typically, the N searchers are assumed to be homogeneous and noninteracting, in which case $\tau_1, \ldots, \tau_N$ are independent and identically distributed (iid). More generally, some systems depend on the kth fastest FPT for $k \in \{1, \ldots, N\}$,

$$T_{k,N} := \min\left\{\{\tau_1, \ldots, \tau_N\} \backslash \cup_{j=1}^{k-1} \{T_{j,N}\}\right\},$$

where $T_{1,N} := T_N$. For searchers which move by diffusion through a continuous state space, many works have studied extreme FPTs [22–35]. However, much less is known about extreme FPTs for processes on discrete state spaces.

In this paper, we study extreme FPTs for $N \gg 1$ searchers on a network. We find explicit and rigorous approximations of the full probability distribution and all the moments of T_N for large N. We obtain these results by proving that a certain rescaling of T_N converges in distribution to a Weibull random variable. In addition, we generalize these results to the kth fastest FPT, $T_{k,N}$, for any $k \ll N$.

In our model, each searcher moves according to a continuous-time random walk (CTRW). To describe how such a CTRW searcher moves through a network, one must specify (i) where a searcher moves and (ii) when a searcher moves. For (i), we assume that the discrete-time process obtained by observing the sequence of nodes visited by the searcher is a discrete-time Markov chain. For (ii), we first assume that the waiting time between moves is exponentially distributed. In this case, the searcher is a continuous-time Markov chain (CTMC). We then generalize (ii) by allowing the waiting time to come from a general family of probability distributions.

Our results show that extreme FPTs are much faster than single FPTs,

$$T_N \ll \tau, \qquad (2)$$

less variable than single FPTs, and independent of much of the details of the network. In particular, while a typical searcher explores much of the network before finding the target, the fastest searcher follows a certain geodesic path(s) from its starting location to the target. See Fig. 1 for an illustration. Indeed, our explicit formulas for the distribution and moments of extreme FPTs depend only on the parameters along this geodesic path(s). Furthermore, our moment formulas are accompanied by rigorous convergence rates, which thus allow one to estimate when a particular system is in the extreme regime in (2). That is, while it is quite intuitive that $T_N \ll \tau$ for sufficiently large N, our analysis determines quantitatively what constitutes "large N" and estimates T_N in this large N regime.

We use our results to investigate the well-studied problem of extreme FPTs of diffusive searchers. This analysis reveals some potential pitfalls in modeling diffusive or subdiffusive processes involving extreme statistics. Specifically, we show that the many searcher limit ($N \to \infty$) does not commute with the diffusion limit for random walks. That is, extreme FPTs differ drastically between "diffusion" modeled with a continuous state space and "diffusion" modeled with a discrete state space.
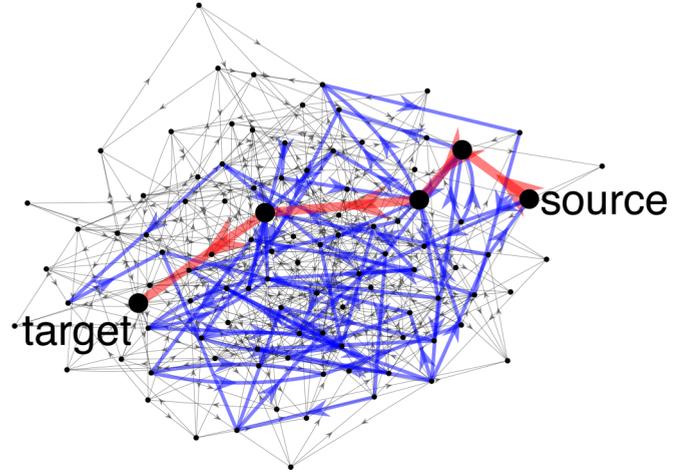

FIG. 1. While a typical searcher (blue trajectory) wanders around the network before reaching the target, the fastest searcher (red trajectory) out of many follows a certain geodesic path(s) from its initial position (source) to the target.

In addition, we prove qualitatively similar results for so-called *mortal* searchers [27,36–43]. Mortal searchers cannot search for the target indefinitely, but rather may be inactivated (degrade, die, evanesce, etc.) before finding the target. We find formulas for the moments of the FPT of a single searcher who is conditioned to find the target before a fast inactivation time. Similar to our results for extreme FPTs and the results in Refs. [42,43] for diffusive mortal searchers, we find that such conditional searchers take a direct route the target.

The rest of the paper is organized as follows. In Sec. II we analyze the case that the searchers are CTMCs, which means the waiting times are exponentially distributed. In Sec. III we extend our analysis to more general waiting time distributions. In Sec. IV we compare our results to numerical simulations on complex networks. In Sec. V we apply our results to extreme FPTs of diffusive and subdiffusive searchers. In Sec. VI we consider mortal searchers. We conclude by discussing our results in the context of several related works.

II. EXPONENTIAL WAITING TIMES

A. Random walk setup

Let $X = \{X(t)\}_{t \geq 0}$ be a CTMC on a finite or countably infinite state space I. The process X is a single searcher (random walker), and the state space I is the nodes (vertices) of the network (graph). There is a directed edge from $i \in I$ to $j \in I$ if X can jump directly from i to j. We refer to the time it takes X to jump directly from i to j as the waiting time. Since X is a CTMC, such waiting times are exponentially distributed.

The dynamics of X are described by its infinitesimal generator matrix [44],

$$Q = \{q(i, j)\}_{i,j \in I}.$$

The off-diagonal entries of Q are non-negative,

$$q(i, j) \geq 0, \quad i \neq j,$$

and give the rate that X jumps from state $i \in I$ to state $j \in I$. The diagonal entries of Q are nonpositive,

$$q(i, i) \leqslant 0, \quad i \in I,$$

and are chosen so that Q has zero row sums,

$$\sum_{j \in I} q(i, j) = 0, \quad i \in I. \tag{3}$$

It is convenient to define

$$q(i) := -q(i, i) \geqslant 0, \quad i \in I,$$

which is the total rate that X leaves state $i \in I$ (regardless of the state that X jumps to). We assume that

$$\sup_{i \in I} q(i) < \infty,$$

which ensures that X cannot take infinitely many jumps in finite time.

B. Single FPTs

Define the FPT of X to some target set $I_{\text{target}} \subset I$,

$$\tau := \inf\{t > 0 : X(t) \in I_{\text{target}}\}. \tag{4}$$

Denote the initial distribution of X by

$$\rho = \{\rho(i)\}_{i \in I} = \{\mathbb{P}(X(0) = i)\}_{i \in I}.$$

To avoid trivial cases, we assume that X cannot start directly on the target, which means

$$I_{\text{target}} \cap \text{supp}(\rho) = \varnothing, \tag{5}$$

where $\text{supp}(\rho) \subset I$ denotes the support of the distribution ρ,

$$\text{supp}(\rho) := \{i \in I : \rho(i) > 0\}.$$

We refer to τ as a single FPT, since it is the FPT of a given single searcher X. We are interested in studying the fastest (extreme) FPT out of $N \gg 1$ searchers,

$$T_N := \min\{\tau_1, \ldots, \tau_N\},$$

where $\tau_1, \ldots, \tau_N$ are N iid realizations of τ. Since $\tau_1, \ldots, \tau_N$ are iid, it is immediate that the distribution of T_N is

$$\mathbb{P}(T_N > t) = (\mathbb{P}(\tau > t))^N. \tag{6}$$

Since $\mathbb{P}(\tau > t)$ decreases monotonically in $t > 0$, it is intuitively clear from (6) that the large N distribution of T_N is determined by the short-time distribution of τ. In this subsection, we find this short-time distribution. In particular, we prove in Proposition 1 below that

$$\mathbb{P}(\tau \leqslant t) \sim \frac{\Lambda}{d!} t^d \quad \text{as } t \to 0+, \tag{7}$$

where (i) $d \geqslant 1$ is the smallest number of jumps that X must take to reach I_{target} and (ii) $\Lambda > 0$ is a sum of the products of the jump rates along the shortest paths from $\text{supp}(\rho)$ to I_{target} (where the terms in the sum are weighted according to ρ). In the remainder of this subsection, we make (7) precise.

Define a path $\mathcal{P}$ of length $d \in \mathbb{Z}_{\geqslant 0}$ from a state $i_0 \in I$ to a state $i_d \in I$ to be a sequence of $d + 1$ states in I,

$$\mathcal{P} = (\mathcal{P}(0), \ldots, \mathcal{P}(d)) = (i_0, i_1, \ldots, i_d) \in I^{d+1}, \tag{8}$$

so that

$$q(\mathcal{P}(k), \mathcal{P}(k+1)) > 0, \quad \text{for } k \in \{0, 1, \ldots, d-1\}. \tag{9}$$

In words, (9) means that there is a strictly positive probability that X may traverse the path $\mathcal{P}$. Naturally, we assume that there is a path from the support of ρ to the target,

$$\bigcup_{d \geqslant 1} \{\mathcal{P} \in I^{d+1} : \mathcal{P}(0) \in \text{supp}(\rho), \mathcal{P}(d) \in I_{\text{target}}\} \neq \varnothing. \tag{10}$$

If (10) is violated, then $\tau = T_N = \infty$ almost surely and the problem is trivial.

For a path $\mathcal{P} \in I^{d+1}$, define $\lambda(\mathcal{P})$ to be the product of the rates along the path,

$$\lambda(\mathcal{P}) := \prod_{i=0}^{d-1} q(\mathcal{P}(i), \mathcal{P}(i+1)) > 0. \tag{11}$$

Let $d_{\min}(I_0, I_1) \in \mathbb{Z}_{\geqslant 0}$ denote the length of the geodesic path from a set of nodes $I_0 \subset I$ to another set of nodes $I_1 \subset I$,

$$d_{\min}(I_0, I_1) := \inf\{d : \mathcal{P} \in I^{d+1}, \mathcal{P}(0) \in I_0, \mathcal{P}(d) \in I_1\}. \tag{12}$$

In words, $d_{\min}(I_0, I_1)$ is the smallest number of jumps required for X to move from I_0 to I_1.

Define the set of all paths from I_0 to I_1 with the minimum length $d_{\min}(I_0, I_1)$ in (12),

$$\begin{aligned} & \mathcal{S}(I_0, I_1) \\ & := \{\mathcal{P} \in I^{d+1} : \mathcal{P}(0) \in I_0, \mathcal{P}(d) \in I_1, d = d_{\min}(I_0, I_1)\}. \end{aligned} \tag{13}$$

Define

$$\Lambda(\rho, I_1) := \sum_{\mathcal{P} \in \mathcal{S}(\text{supp}(\rho), I_1)} \rho(\mathcal{P}(0)) \lambda(\mathcal{P}). \tag{14}$$

The quantity $\Lambda(\rho, I_1)$ is easiest to understand by first considering the case that $\rho(i_0) = 1$ for some $i_0 \in I$ [meaning $\text{supp}(\rho) = i_0 = X(0)$ almost surely]. In this case, if there is a unique path with the minimum number of jumps $d_{\min}(i_0, I_1)$, then $\Lambda(\rho, I_1)$ is simply the product of the jump rates along this geodesic path [$\lambda(\mathcal{P})$ in (11)]. If there are multiple such geodesic paths, then $\Lambda(\rho, I_1)$ sums the products of the jump rates along these paths. Finally, if the support of ρ is not concentrated at a single point, then $\Lambda(\rho, I_1)$ simply sums the products of the jump rates along all the geodesic paths, where the sum is weighted according to the initial distribution ρ.

With these definitions in place, we can now give the short-time behavior of the distribution of τ. Throughout this paper,

$$\text{``} f \sim g \text{'' means } f/g \to 1.$$

Proposition 1. We have that

$$\mathbb{P}(\tau \leqslant t) \sim \frac{\Lambda}{d!} t^d \quad \text{as } t \to 0+,$$

where

$$d = d_{\min}(\text{supp}(\rho), I_{\text{target}}) \in \mathbb{Z}_{> 0},$$

$$\Lambda = \Lambda(\rho, I_{\text{target}}) > 0.$$

The proof of Proposition 1, as well as the proofs of the theorems and propositions below, are given in the Appendix.

C. Fastest FPT

Having determined the short-time distribution of a single FPT τ in Proposition 1, we now determine the large N distribution and moments of the fastest FPT T_N. In Theorem 1 below, we prove that a certain rescaling of T_N converges in distribution to a Weibull random variable. Before stating the theorem, we recall two requisite definitions.

Definition 1. A sequence of random variables $\{Z_N\}_{N \geqslant 1}$ *converges in distribution* to a random variable Z if

$$\mathbb{P}(Z_N \leqslant z) \to \mathbb{P}(Z \leqslant z) \quad \text{as } N \to \infty,$$

for all points $z \in \mathbb{R}$ such that $F(z) := \mathbb{P}(Z \leqslant z)$ is continuous. If this holds, then we write

$$Z_N \to_{\mathrm{d}} Z \quad \text{as } N \to \infty.$$

Definition 2. A random variable $Z \geqslant 0$ has a *Weibull distribution* with scale parameter $t > 0$ and shape parameter $d > 0$ if

$$\mathbb{P}(Z > z) = \exp(-(z/t)^d), \quad z \geqslant 0. \tag{15}$$

If (15) holds, then we write

$$Z =_{\mathrm{d}} \text{Weibull}(t, d).$$

Theorem 1. Let $d \geqslant 1$ and $\Lambda > 0$ be as in Proposition 1 and define

$$A = \frac{\Lambda}{d!} > 0.$$

The following rescaling of T_N converges in distribution to a Weibull random variable:

$$(AN)^{1/d} T_N \to_{\mathrm{d}} \text{Weibull}(1, d) \quad \text{as } N \to \infty. \tag{16}$$

Suppose further that

$$\mathbb{E}[T_N] < \infty \quad \text{for some } N \geqslant 1.$$

Then for each moment $m \in (0, \infty)$, we have that

$$\mathbb{E}[(T_N)^m] \sim \frac{\Gamma(1 + m/d)}{(AN)^{m/d}} \quad \text{as } N \to \infty. \tag{17}$$

The convergence in (16) means roughly that the distribution of T_N for large N is given by

$$T_N \approx_{\mathrm{d}} \text{Weibull}((AN)^{-1/d}, d). \tag{18}$$

Further, the general formula for the mth moment in (17) means that as $N \to \infty$ the mean and variance are

$$\mathbb{E}[T_N] \sim \frac{\Gamma\left(1 + \frac{1}{d}\right)}{(AN)^{1/d}},$$

$$\text{Variance}(T_N) \sim \frac{\Gamma\left(1 + \frac{2}{d}\right) - \left(\Gamma\left(1 + \frac{1}{d}\right)\right)^2}{(AN)^{2/d}}. \tag{19}$$

Compared to a single FPT, Theorem 1 implies that extreme FPTs are (i) faster, (ii) less variable, and (iii) less affected by the size, structure, and details of the network. To see points (i) and (ii), note the vanishing mean and variance in (19) [which is implied by the vanishing moments in (17)]. To see point (iii), notice that Theorem 1 implies that the limiting distribution of T_N is completely determined by the parameters N, Λ, and d. In particular, the only network parameters that enter into the large N distribution of T_N are along the geodesic
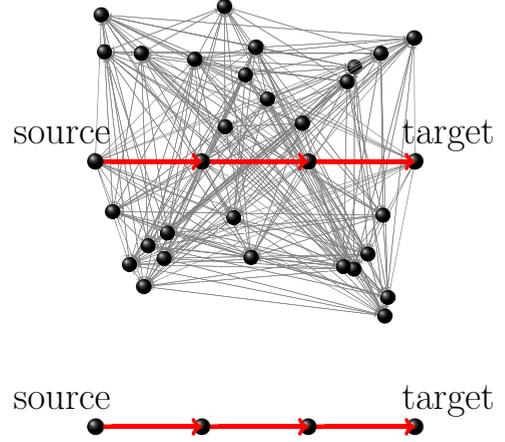

FIG. 2. The fastest searcher out of $N \gg 1$ searchers moving through the top network takes the geodesic path from initial position (source) to the target, which is highlighted in red. Theorems 1 and 2 imply that the statistics of the extreme FPTs depend only on network parameters along this geodesic path. Therefore, deleting the nodes outside this geodesic path does not affect the extreme FPTs. Hence, the complicated top network and the simple bottom network have the same extreme FPTs for large N (assuming the rates along the geodesic path are identical in the top and bottom networks).

path(s) from the initial distribution to the target. Therefore, if there are many searchers ($N \gg 1$), the distribution of T_N is unaffected by changes to the network outside this geodesic path(s). This is illustrated in Fig. 2, which depicts two vastly different networks that nonetheless have the same extreme FPT distributions.

In addition, (17) allows us to estimate when T_N is in the extreme Weibull regime of Theorem 1. In particular, we are assured by (17) that T_N is in this large N regime for all moments $m \geqslant 1$ if $(AN)^{-1/d}$ is much smaller than the other timescales in the problem, which means

$$N \gg \frac{(\sup_{i \in I} q(i))^d}{A} = \frac{d!(\sup_{i \in I} q(i))^d}{\Lambda},$$

since $\sup_{i \in I} q(i) < \infty$ is the fastest rate in the problem.

D. kth fastest FPT

We now generalize Theorem 1 on the fastest FPT to the kth fastest FPT,

$$T_{k,N} := \min \left\{ \{\tau_1, \ldots, \tau_N\} \backslash \cup_{j=1}^{k-1} \{T_{j,N}\} \right\},$$

where $T_{1,N} := T_N$. The large N distribution of $T_{k,N}$ is described in terms of a generalized gamma random variable.

Definition 3. A random variable $X \geqslant 0$ has a *generalized gamma distribution* with parameters $t > 0$, $d > 0$, $k > 0$ if

$$\mathbb{P}(Z > z) = \frac{\Gamma(k, (z/t)^d)}{\Gamma(k)}, \quad z \geqslant 0, \tag{20}$$

where $\Gamma(a, z) := \int_z^\infty u^{a-1} e^{-u} \, du$ denotes the upper incomplete gamma function. If (20) holds, then we write

$$Z =_{\mathrm{d}} \text{gen}\Gamma(t, d, k).$$

Theorem 2. Fix $k \geqslant 1$ and let A be as in Theorem 1. The following rescaling of $T_{k,N}$ converges in distribution to a generalized gamma random variable,

$$(AN)^{1/d} T_{k,N} \to_d \text{gen}\Gamma(1, d, k) \quad \text{as } N \to \infty.$$

Suppose further that

$$\mathbb{E}[T_N] < \infty \quad \text{for some } N \geqslant 1.$$

Then for each moment $m \in (0, \infty)$, we have that

$$\mathbb{E}[(T_{k,N})^m] \sim \frac{\Gamma(k + m/d)/\Gamma(k)}{(AN)^{m/d}} \quad \text{as } N \to \infty. \quad (21)$$

Similar to (18), Theorem 2 means roughly that the distribution of $T_{k,N}$ for large N is

$$T_{k,N} \approx_d \text{gen}\Gamma((AN)^{-1/d}, d, k).$$

Further, the general formula for the mth moment in (21) means that as $N \to \infty$ the mean and variance are

$$\mathbb{E}[T_{k,N}] \sim \frac{\Gamma\left(k + \frac{1}{d}\right)/\Gamma(k)}{(AN)^{1/d}},$$

$$\text{Variance}(T_{k,N}) \sim \frac{\Gamma\left(k + \frac{2}{d}\right)/\Gamma(k) - \left(\Gamma\left(k + \frac{1}{d}\right)/\Gamma(k)\right)^2}{(AN)^{2/d}}.$$

III. GENERAL WAITING TIMES

A. Random walk setup

In the previous section, we took the random walkers to be CTMCs. From a modeling perspective, there are three key restrictions implied by this assumption. First, the waiting times are always exponentially distributed. Second, the waiting times depend only on the current state. That is, the waiting time to jump from node i to node j depends only on i and not on the destination j [the waiting time is exponential with rate $q(i)$ even though the "jump rate" is $q(i, j)$; see below]. Naturally, in many applications the time it takes to move depends on the destination. Third, the waiting times may be arbitrarily short. That is, for any small time $\varepsilon > 0$, there is a strictly positive probability that the searcher will jump from i to j in a time less than ε [as long as $q(i, j) > 0$]. In this section, we remove these three restrictions.

Let $X = \{X(t)\}_{t \geqslant 0}$ be a continuous-time stochastic process on a discrete state space I (as a technical point, we assume paths of X are continuous from the right). In contrast to Sec. II, we assume for simplicity that I is finite. Informally, we suppose that the CTRW X walks on the network I in the following manner. From a state $i \in I$, the walker chooses its next state according to a probability distribution that depends on only its current state i. Then, having chosen that the next state is some $j \in I$, the walker waits at its current state until time $S > 0$, where S is chosen according to a probability distribution that may depend on both the current state i and the next state j.

More precisely, let $J(0), J(1), \ldots$ be the jump times of X, which are defined by [44]

$$J(0) = 0,$$

$$J(n + 1) = \inf\{t \geqslant J(n) : X(t) \neq X(J(n))\}, \quad n \geqslant 0.$$

Further, define the waiting times $S(1), S(2), \ldots$ by

$$S(n) := \begin{cases} J(n) - J(n-1) & \text{if } J(n-1) < \infty, \\ \infty & \text{otherwise,} \end{cases} \quad n \geqslant 1.$$

In words, the nth jump of X happens at time $t = J(n)$, and X waits at its new state for time $S(n + 1)$.

Assume that the discrete-time process obtained by observing X only at the jump times,

$$Y(n) := X(J(n)), \quad n \geqslant 0, \quad (22)$$

is a discrete-time Markov chain. Further, assume that for each $n \geqslant 1$, conditional on $Y(0), \ldots, Y(n)$, the waiting times $S(1), \ldots, S(n)$ are independent random variables with

$$\mathbb{P}(S(m) \leqslant t) = F_{Y(m-1),Y(m)}(t), \quad t \in \mathbb{R}, \quad (23)$$

where $\{F_{i,j}(t)\}_{i,j \in I}$ are a given set of cumulative distribution functions. In addition, assume that each $F_{i,j}(t)$ satisfies

$$F_{i,j}(t) = 0, \quad t \leqslant t_0(i, j), \quad (24)$$

$$F_{i,j}(t) \sim \lambda(i, j)(t - t_0(i, j))^r, \quad \text{as } t \to t_0(i, j)+, \quad (25)$$

where $t_0(i, j) \geqslant 0$, $\lambda(i, j) > 0$, and $r > 0$. Assume that

$$\sup_{i,j \in I} \lambda(i, j) < \infty,$$

which ensures that X cannot take infinitely many jumps in finite time (that is, X is not explosive).

The assumption in (23) means that the waiting time $S(n)$ from state $Y(n - 1)$ to state $Y(n)$ can depend on both $Y(n - 1)$ and $Y(n)$. The assumption in (24) means that $t_0(i, j) \geqslant 0$ is the fastest possible waiting time from state i to state j. From a modeling perspective, the benefit of this assumption is that it allows one to ensure that waiting times cannot be arbitrarily small by setting $t_0(i, j) > 0$. The assumption in (25) describes the waiting time distribution near the fastest waiting time $t_0(i, j)$.

We note that if X is merely a CTMC as in Sec. II, then

$$F_{i,j}(t) = 1 - e^{-q(i)t}, \quad t \geqslant t_0(i, j) = 0,$$
$$\lambda(i, j) = q(i),$$
$$r = 1. \quad (26)$$

In particular, notice that $F_{i,j}(t)$ in (26) depends only on i for the case of a CTMC.

B. Single FPTs

To describe the short-time distribution of the FPT τ in (4) for this generalized process X, we must generalize our definitions in (8)–(14). First, let

$$\Pi = \{\pi(i, j)\}_{i,j \in I}$$

be the stochastic matrix governing the discrete-time process $Y(n)$ in (22) [44]. In particular, $\pi(i, j)$ is the probability that Y jumps from i to j. Define a path $\mathcal{P}$ of length $d \in \mathbb{Z}_{\geqslant 0}$ from a state $i_0 \in I$ to a state $i_d \in I$ to be a sequence of $d + 1$ states in I,

$$\mathcal{P} = (\mathcal{P}(0), \ldots, \mathcal{P}(d)) = (i_0, i_1, \ldots, i_d) \in I^{d+1}, \quad (27)$$

so that

$$\pi(\mathcal{P}(k), \mathcal{P}(k+1)) > 0, \quad \text{for } k \in \{0, 1, \ldots, d-1\}. \quad (28)$$

Note that (27) and (28) generalize the definition of a path in (8) and (9) since if X is a CTMC as in Sec. II, then [44]

$$\pi(i, j) = \begin{cases} q(i, j)/q(i) & \text{if } j \neq i \text{ and } q(i) \neq 0 \\ 0 & \text{if } j \neq i \text{ and } q(i) = 0 \end{cases},$$

$$\pi(i, i) = \begin{cases} 0 & \text{if } q(i) \neq 0 \\ 1 & \text{if } q(i) = 0 \end{cases}. \quad (29)$$

As in Sec. II, we assume of course that there is a path from the support of ρ to the target [see (10)]. We also assume (5), which means the searcher cannot start directly on the target.

For a path $\mathcal{P} \in I^{d+1}$, define $\lambda(\mathcal{P})$ to be the product of the $\pi(i, j)$'s and $\lambda(i, j)$'s in (25) along the path,

$$\lambda(\mathcal{P}) := \prod_{i=0}^{d-1} \pi(\mathcal{P}(i), \mathcal{P}(i+1))\lambda(\mathcal{P}(i), \mathcal{P}(i+1)) > 0. \quad (30)$$

Note that (26) and (29) imply that (30) generalizes (11). In addition, for a path $\mathcal{P} \in I^{d+1}$, define $t_0(\mathcal{P})$ to be the sum of the $t_0(i, j)$'s in (25) along the path,

$$t_0(\mathcal{P}) := \sum_{i=0}^{d-1} t_0(\mathcal{P}(i), \mathcal{P}(i+1)) \geq 0.$$

In words, $t_0(\mathcal{P})$ is the shortest possible time required to traverse the path $\mathcal{P}$. Taking the infimum over paths, define the shortest possible time to reach a set $I_1 \subset I$ starting from a set $I_0 \subset I$,

$$t_{\min}(I_0, I_1)$$
$$:= \inf\{t_0(\mathcal{P}) : \mathcal{P} \in I^{d+1}, \mathcal{P}(0) \in I_0, \mathcal{P}(d) \in I_1\} \geq 0.$$

Next, define the smallest number of jumps required to reach I_1 from I_0 if the searcher traverses a path $\mathcal{P}$ with the minimum required time $t_0(\mathcal{P}) = t_{\min}(I_0, I_1)$,

$$d_{\min}(I_0, I_1)$$
$$:= \inf\{d : \mathcal{P}(0) \in I_0, \mathcal{P}(d) \in I_1, t_0(\mathcal{P}) = t_{\min}(I_0, I_1)\}.$$

Further, define

$$\mathcal{S}(I_0, I_1) := \{\mathcal{P} \in I^{d+1} : \mathcal{P}(0) \in I_0, \mathcal{P}(d) \in I_1,$$
$$t_0(\mathcal{P}) = t_{\min}(I_0, I_1), d = d_{\min}(I_0, I_1)\}.$$

In words, $\mathcal{S}(I_0, I_1)$ are the paths $\mathcal{P}$ going from I_0 to I_1 which (i) have the minimum time and (ii) have the minimum number of jumps out of the paths which have the minimum time. Define

$$\Lambda(\rho, I_1) := \sum_{\mathcal{P} \in \mathcal{S}(\text{supp}(\rho), I_1)} \rho(\mathcal{P}(0))\lambda(\mathcal{P}).$$

It is immediate that these definitions generalize the definitions in Sec. II since $t_0(\mathcal{P}) = 0$ for every path in the case that X is a CTMC.

Proposition 2. We have that

$$\mathbb{P}(\tau \leq t_{\min} + t) \sim \frac{(\Gamma(r+1))^d}{\Gamma(dr+1)} \Lambda t^d \quad \text{as } t \to 0+,$$

where $r > 0$ is in (25) and

$$t_{\min} = t_{\min}(\text{supp}(\rho), I_{\text{target}}) \geq 0,$$
$$d = d_{\min}(\text{supp}(\rho), I_{\text{target}}) \in \mathbb{Z}_{>0},$$
$$\Lambda = \Lambda(\rho, I_{\text{target}}) > 0.$$

C. Extreme FPTs

Having determined the short-time distribution of a single FPT τ in Proposition 2, we now determine the distribution and moments of the fastest FPT, T_N, and the kth fastest FPT, $T_{k,N}$, out of $N \gg 1$ iid realizations of τ.

Theorem 3. Let $t_{\min} \geq 0, d \geq 1, r > 0$, and $\Lambda > 0$ be as in Proposition 2 and define

$$A = \frac{(\Gamma(r+1))^d}{\Gamma(dr+1)} \Lambda > 0.$$

The following rescaling of $T_N - t_{\min}$ converges in distribution to a Weibull random variable,

$$(AN)^{1/d}(T_N - t_{\min}) \to_d \text{Weibull}(1, d) \quad \text{as N} \to \infty.$$

Suppose further that

$$\mathbb{E}[T_N] < \infty \quad \text{for some } N \geq 1.$$

Then for each moment $m \in (0, \infty)$, we have that

$$\mathbb{E}[(T_N - t_{\min})^m] \sim \frac{\Gamma(1 + m/d)}{(AN)^{m/d}} \quad \text{as } N \to \infty.$$

Theorem 4. Fix $k \geq 1$ and let A be as in Theorem 3. The following rescaling of $T_{k,N} - t_{\min}$ converges in distribution to a generalized gamma random variable,

$$(AN)^{1/d}(T_{k,N} - t_{\min}) \to_d \text{gen}\Gamma(1, d, k) \quad \text{as N} \to \infty.$$

Suppose further that

$$\mathbb{E}[T_N] < \infty \quad \text{for some } N \geq 1.$$

Then for each moment $m \in (0, \infty)$, we have that

$$\mathbb{E}[(T_{k,N} - t_{\min})^m] \sim \frac{\Gamma(k + m/d)/\Gamma(k)}{(AN)^{m/d}} \quad \text{as } N \to \infty.$$

IV. NUMERICAL SIMULATIONS

In this section, we compare the results of our analysis to numerical simulations on complex networks. We consider the setup of Sec. II in which each searcher moves according to a CTMC.

To create the CTMC, we create a graph by randomly connecting $V = |I| \gg 1$ vertices by E directed edges (we construct the graph so that $E \approx 5V$). We then assign jump rates to each directed edge independently according to a uniform distribution. More precisely, if the CTMC has infinitesimal generator matrix $Q = \{q(i, j)\}_{i,j \in I}$, then the diagonal entries, $q(i, i) \leq 0$ are chosen so that Q has zero row sums [see (3)], and the off-diagonal entries, $q(i, j) \geq 0$ with $i \neq j$, are

$$q(i, j) = \begin{cases} U_{i,j} & \text{if there is a directed edge from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases},$$

where $\{U_{i,j}\}_{i,j \in I}$ are independent uniform random variables on [0,1].
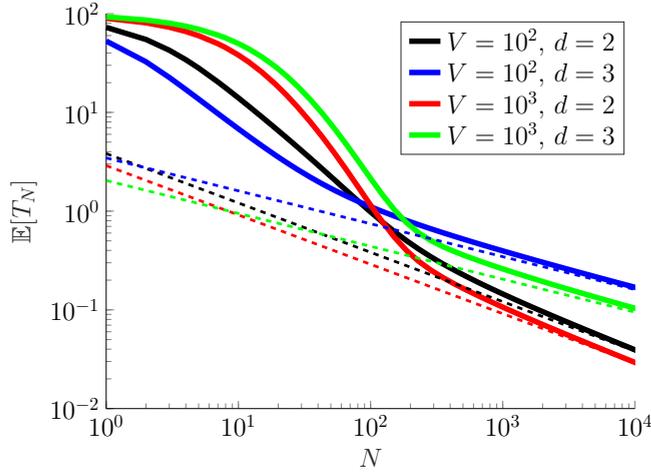
FIG. 3. Mean fastest FPT, $\mathbb{E}[T_N]$, as a function of the number of searchers, N. The different colored solid curves are $\mathbb{E}[T_N]$ computed numerically for different random graphs (different CTMCs). These curves approach the theoretical values [see Theorem 1 and (34)], which are the dashed lines.

To numerically compute the distribution and mean of the fastest FPT T_N, we need only compute the survival probability $\mathbb{P}(\tau > t)$ of a single FPT τ since

$$\mathbb{P}(T_N > t) = (\mathbb{P}(\tau > t))^N, \tag{31}$$

$$\mathbb{E}[T_N] = \int_0^\infty (\mathbb{P}(\tau > t))^N \, dt. \tag{32}$$

To compute $\mathbb{P}(\tau > t)$, let the target be a single node, $I_{\text{target}} = i_{\text{target}} \in I$, and let $\widetilde{Q}$ denote the matrix obtained by deleting the row and column in Q corresponding to i_{target}. Similarly, for an initial distribution ρ, let $\widetilde{\rho}$ denote the vector obtained by deleting the entry in ρ corresponding to i_{target}. Then, $\mathbb{P}(\tau > t)$ is given by the sum of the entries in the vector $e^{\widetilde{Q}^\top t}\widetilde{\rho}$, where $\widetilde{Q}^\top$ denotes the transpose of $\widetilde{Q}$ and $e^{\widetilde{Q}^\top t}$ denotes the matrix exponential [45]. In particular, we can write $\mathbb{P}(\tau > t)$ as the dot product,

$$\mathbb{P}(\tau > t) = \mathbf{1} \cdot e^{\widetilde{Q}^\top t}\widetilde{\rho}, \tag{33}$$

where $\mathbf{1} \in \mathbb{R}^{V-1}$ is the vector of all ones.

In Fig. 3 we plot the mean fastest FPT, $\mathbb{E}[T_N]$, as a function of the number of searchers, N, for different values of the number of vertices V and the shortest distance d from the starting location to the target state. The solid curves are $\mathbb{E}[T_N]$ computed from (32), with $\mathbb{P}(\tau > t)$ computed from (33). The dashed lines are the large N formula for $\mathbb{E}[T_N]$ found in Theorem 1, namely,

$$\frac{\Gamma(1 + 1/d)}{(AN)^{1/d}}. \tag{34}$$

In agreement with the theory, the solid curves in Fig. 3 approach the corresponding dashed lines as N increases. In particular, this plot illustrates that the MFPT of a single searcher ($\mathbb{E}[T_1] = \mathbb{E}[\tau] \approx 10^2$) is much slower than the MFPT of the fastest searcher out of many searchers ($\mathbb{E}[T_N] \ll \mathbb{E}[\tau]$ if $N \gg 10^2$).
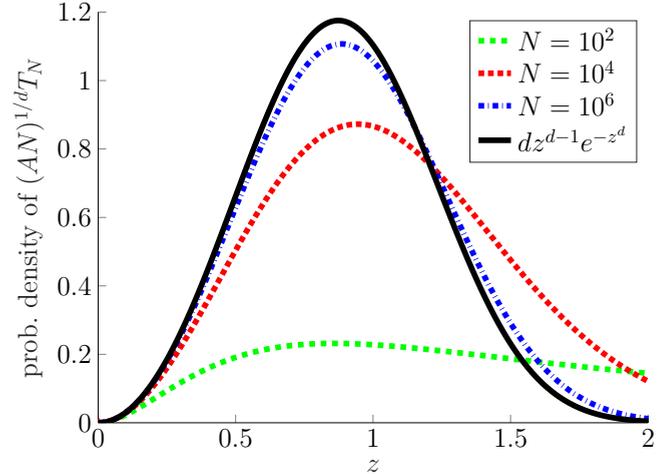

FIG. 4. Probability density of the rescaled fastest FPT, $(AN)^{1/d}T_N$, for different values of the number of searchers, N. The black solid curve is the probability density of a Weibull random variable with unit scale parameter and shape parameter d. In this plot, the random graph has $V = 10^3$ vertices (states in the CTMC), and the shortest distance from the starting state to the target is $d = 3$.

In addition to the moments of T_N, Theorem 1 gives the full probability distribution of T_N for large N. We illustrate this convergence in Fig. 4 by plotting the probability density of the rescaled fastest FPT, $(AN)^{1/d}T_N$, for different values of N. The probability density of $(AN)^{1/d}T_N$ is computed from (31). In this plot, the graph has $V = 10^3$ vertices (states for the Markov chain) and the shortest distance from the starting location to the target is $d = 3$. In agreement with the theory, the probability density of $(AN)^{1/d}T_N$ approaches the density of a Weibull random variable with unit scale parameter and shape parameter d (namely, the limiting density is $dz^{d-1}e^{-z^d}$).

V. (SUB)DIFFUSIVE SEARCHERS

A. Diffusion

In this subsection, we compare our results to extreme FPTs for continuous state space diffusion processes, which have been studied extensively [22–35]. Let $X^{\text{diff}} = \{X^{\text{diff}}(t)\}_{t \geqslant 0}$ be a one-dimensional continuous state space diffusion process starting at the origin with diffusivity $D > 0$. That is, suppose X^{diff} satisfies the stochastic differential equation,

$$dX^{\text{diff}} = \sqrt{2D} \, dW, \quad X(0) = 0,$$

where $W = \{W(t)\}_{t \geqslant 0}$ is a standard Brownian motion. Hence, the probability density $p(x, t)$ that $X(t) = x$ satisfies the Fokker-Planck equation,

$$\frac{\partial}{\partial t}p = D\frac{\partial^2}{\partial x^2}p, \quad x \in \mathbb{R}, \, t > 0, \tag{35}$$

$$p(x, 0) = \delta(x).$$

Let τ^{diff} be the first time that X^{diff} escapes the interval $(-L, L)$,

$$\tau^{\text{diff}} := \inf\{t > 0 : X^{\text{diff}}(t) \notin (-L, L)\}, \tag{36}$$

and define the extreme FPT,

$$T_N^{\text{diff}} := \min\left\{\tau_1^{\text{diff}}, \dots, \tau_N^{\text{diff}}\right\}, \qquad (37)$$

where $\tau_1^{\text{diff}}, \dots, \tau_N^{\text{diff}}$ are N iid realizations of τ^{diff}. It is well known that [23]

$$\mathbb{E}\left[T_N^{\text{diff}}\right] \sim \frac{L^2}{4D \ln N}, \quad \text{as } N \to \infty. \qquad (38)$$

Indeed, the asymptotic behavior in (38) holds in much greater generality, including for diffusion processes on d-dimensional manifolds with space-dependent diffusivities and force fields [22].

It is interesting to compare the behavior in (38) for diffusive searchers to the behavior we found in Theorem 1 for CTMCs. Suppose we discretize space with step size

$$\Delta x = \frac{L}{d} \ll L, \qquad (39)$$

where d is a large natural number. Let $\{X^{\text{ctmc}}(t)\}_{t \geqslant 0}$ be the CTMC which takes values in the one-dimensional network,

$$I := \{i \Delta x\}_{i \in \mathbb{Z}} = \{\dots, -\Delta x, 0, \Delta x, \dots\}, \qquad (40)$$

and has jump rates (using the notation of Sec. II),

$$q(i, i \pm 1) = \frac{q(i)}{2} = \frac{D}{(\Delta x)^2} = \frac{D}{L^2} d^2 > 0, \quad i \in \mathbb{Z}.$$

Assume $X^{\text{ctmc}}(0) = 0$. Depending on the context, X^{diff} may be viewed as an approximation of X^{ctmc}, or vice versa. The correspondence between X^{diff} and X^{ctmc} is perhaps most easily seen by noticing that if we use a centered, second-order finite difference approximation for the spatial derivative in the Fokker-Planck equation (35) for X^{diff}, then we obtain the master equation (Kolmogorov forward equation) for X^{ctmc}.

Define τ^{ctmc} and T_N^{ctmc} analogously to τ^{diff} and T_N^{diff},

$$\begin{aligned} \tau^{\text{ctmc}} &:= \inf\{t > 0 : X^{\text{ctmc}}(t) \notin \{i \Delta x\}_{|i| < d}\}, \\ T_N^{\text{ctmc}} &:= \min\left\{\tau_1^{\text{ctmc}}, \dots, \tau_N^{\text{ctmc}}\right\}, \end{aligned} \qquad (41)$$

where $\tau_1^{\text{ctmc}}, \dots, \tau_N^{\text{ctmc}}$ are N iid realization of τ^{ctmc}. Theorem 1 above implies that

$$\mathbb{E}\left[T_N^{\text{ctmc}}\right] \sim \frac{L^2}{D} \frac{f(d)}{N^{1/d}}, \quad \text{as } N \to \infty, \qquad (42)$$

where

$$f(d) := \frac{(d!)^{1/d} \Gamma(1 + 1/d)}{d^2} \sim \frac{1}{de}, \quad \text{as } d \to \infty.$$

Hence, while the distributions of $X^{\text{diff}}(t)$ and $X^{\text{ctmc}}(t)$ can be made close for any fixed $t \geqslant 0$ by taking d large, we see from (38) and (42) that the extreme FPTs of X^{diff} and X^{ctmc} are quite different for any $d \geqslant 1$.

Put another way, this shows that the diffusion limit, $d \to \infty$, and the many searcher limit, $N \to \infty$, of T_N^{ctmc} do not commute. From a modeling perspective, this means that care must be taken in choosing a model of diffusion (spatially continuous X^{diff} versus spatially discrete X^{ctmc}) if the system depends on extreme statistics. See Ref. [46] for an analysis of extreme statistics of diffusion modeled by a piecewise deterministic Markov process (i.e., a velocity jump process).

B. Subdiffusion

In this subsection, we compare our results to extreme FPTs for subdiffusive processes. A subdiffusive process $\{X(t)\}_{t \geqslant 0}$ is defined by a mean-squared displacement that grows sublinearly in time [47],

$$\mathbb{E}[(X(t) - X(0))^2] \propto t^\alpha, \quad \alpha \in (0, 1).$$

A common model for subdiffusion is a certain type of CTRW [48]. In one space dimension, this model is characterized by a jump length probability density function (pdf), $l(x)$, and a waiting time pdf, $w(t)$. In particular, if the searcher lands at some position $Y(n) \in \mathbb{R}$, the searcher waits until a time chosen from $w(t)$, then jumps to a new location $Y(n + 1) = Y(n) + \xi(n + 1)$, where $\xi(n + 1)$ is chosen from $l(x)$. The searcher continues this process indefinitely.

Assume that the jump length pdf $l(x)$ is symmetric about the origin so that the walk is unbiased, and assume that it has finite standard deviation,

$$\Delta x := \sqrt{\int_{-\infty}^\infty x^2 l(x) \, dx} < \infty.$$

In addition, assume that the waiting time pdf has a slow power-law decay,

$$w(t) \sim C_\alpha \left(\frac{\Delta t}{t}\right)^{1+\alpha}, \quad \text{as } t \to \infty, \qquad (43)$$

where $\alpha \in (0, 1)$, for some timescale Δt and some rate $C_\alpha > 0$. Choose $l(x)$ so that $\Delta x = L/d$ as in (39) and choose $w(t)$ so that Δt satisfies

$$(\Delta t)^\alpha = \frac{(\Delta x)^2}{2K_\alpha} = \frac{(L/d)^2}{2K_\alpha}, \qquad (44)$$

where $K_\alpha > 0$ is some fixed generalized diffusivity. Then, in the diffusion limit $d \to \infty$, it is well known that the pdf of the limiting process satisfies the fractional Fokker-Planck equation [48],

$$\frac{\partial}{\partial t} p = {}_0D_t^{1-\alpha} K_\alpha \frac{\partial^2}{\partial x^2} p, \quad x \in \mathbb{R}, \, t > 0, \qquad (45)$$

where ${}_0D_t^{1-\alpha}$ is the fractional derivative of Riemann-Liouville type [49], defined by

$${}_0D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} \, ds.$$

Let $X^{\text{sub}} = \{X^{\text{sub}}(t)\}_{t \geqslant 0}$ denote the subdiffusive process starting at the origin,

$$X^{\text{sub}}(0) = 0,$$

whose pdf satisfies the fractional equation (45) (note that X^{sub} can be constructed as a random time change of X^{diff} [50]). Define τ^{sub} and T_N^{sub} analogously to (36) and (37). It was recently proven [35] that

$$\mathbb{E}\left[T_N^{\text{sub}}\right] \sim \frac{t_\alpha}{(\ln N)^{2/\alpha - 1}} \quad \text{as } N \to \infty, \qquad (46)$$

where $t_\alpha > 0$ is the timescale,

$$t_\alpha := \left(\alpha^\alpha (2 - \alpha)^{2-\alpha} \frac{L^2}{4K_\alpha}\right)^{1/\alpha} > 0.$$

The CTRW leading to the fractional equation (45) can be put in the framework of Sec. III. In particular, consider a process $\{X^{\mathrm{ctrw}}(t)\}_{t \geqslant 0}$ with waiting time pdf $w(t)$ satisfying (43) and (44) and jump length pdf $l(x)$ given by a sum of Dirac delta functions,

$$l(x) = \tfrac{1}{2}\delta(\Delta x - x) + \tfrac{1}{2}\delta(\Delta x + x), \quad x \in \mathbb{R},$$

with $\Delta x = L/d$. In the notation of Sec. III, the state space I is the discrete set in (40), the jump chain follows

$$\pi(i, j) = \begin{cases} 1/2 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F_{i,i\pm 1}(t) = \int_0^t w(t')\, dt', \quad i \in \mathbb{Z}.$$

Suppose that $\lim_{t \to 0+} w(t) = 2\lambda(d) > 0$ for some function $\lambda(\cdot)$ so that $F(t) := F_{i,i\pm 1}(t) \sim 2\lambda(d)t$ as $t \to 0+$. Thus F satisfies (24) and (25) with $r = 1$ and $t_0 = 0$.

Therefore, in the diffusion limit $d \to \infty$, the pdf of the limiting process satisfies (45), and thus the extreme FPTs satisfy (46). That is, if we take $d \to \infty$ first, and then take $N \to \infty$ limit, then we obtain (46). However, Theorem 3 above shows that if we take $N \to \infty$ first for the CTRW X^{ctrw}, then we obtain that the extreme FPTs satisfy

$$\mathbb{E}\big[T_N^{\mathrm{ctrw}}\big] \sim \left[\frac{\Gamma(1 + 1/d)}{(d!)^{1/d}\lambda(d)}\right]\frac{1}{N^{1/d}}, \quad \text{as } N \to \infty, \qquad (47)$$

where T_N^{ctrw} is defined analogously to (41).

Comparing (46) and (47), we again see that the diffusion limit ($d \to \infty$) and the many searcher limit ($N \to \infty$) do not commute. In addition, comparing (42) and (47) shows that the extreme FPTs of the discrete state space diffusive process X^{ctmc} and the discrete state space subdiffusive process X^{ctrw} both decay as $N^{-1/d}$ as $N \to \infty$. In fact,

$$\mathbb{E}\big[T_N^{\mathrm{ctmc}}\big] \sim \mathbb{E}\big[T_N^{\mathrm{ctrw}}\big], \quad \text{as } N \to \infty,$$

if we take $\lambda(d) = (D/L^2)d^2$. Hence, the behavior of extreme statistics is very different in the discrete case ($d < \infty$) compared to the continuum limit ($d = \infty$).

VI. FAST INACTIVATION OF MORTAL WALKERS

Compared to a single FPT, we found in Secs. II and III that extreme FPTs are faster, less variable, and less affected by network size and structure. In essence, considering only the fastest FPTs filters out searchers which deviate from a direct route to the target. It was recently shown in Refs. [42,43] that *fast inactivation* can have a similar effect on FPTs by filtering out slow searchers. These two prior works considered searchers which move by continuous state space diffusion [42,43] or discrete state space diffusion [42]. In this section, we consider fast inactivation for searchers on networks which

move according to a CTMC as in Sec. II or a CTRW as in Sec. III.

Consider a single searcher that can be inactivated (degrade, die, evanesce, etc.) before reaching the target. Such finite lifetime searchers are called "mortal" or "evanescent" and have been widely studied [27,36–43]. Indeed, mortal searchers have been used to model a variety of systems, including inactivation of intracellular signaling molecules [42], sperm cells searching for an egg despite a high mortality rate [27], animals or bacteria foraging for food, extinction of a fluorescent signal in bio-imaging methods, messenger RNA searching for a ribosome, and storage of nuclear waste [41].

Mathematically, in addition to the FPT τ of a single searcher [as in (1)], one introduces an independent and exponentially distributed inactivation time σ with rate $\gamma > 0$,

$$\sigma =_{\mathrm{d}} \text{exponential}(\gamma).$$

Hence, the event $\tau < \sigma$ means that the searcher found the target before it was inactivated, while $\sigma < \tau$ corresponds to the opposite scenario. Consider the mth moment of τ, conditioned that the searcher finds the target before it is inactivated,

$$\mathbb{E}[\tau^m \mid \tau < \sigma] := \frac{\mathbb{E}[\tau^m 1_{\tau < \sigma}]}{\mathbb{P}(\tau < \sigma)}, \qquad (48)$$

where $1_{\tau < \sigma}$ denotes the indicator function,

$$1_{\tau < \sigma} := \begin{cases} 1 & \text{if } \tau < \sigma, \\ 0 & \text{otherwise} \end{cases}.$$

As in Refs. [42,43], we are interested in the behavior of the conditional FPT moments (48) in the limit of fast inactivation, i.e., $\gamma \to \infty$. The following theorem gives this behavior in terms of the short-time behavior of the unconditioned FPT τ. In particular, Theorem 5 is stated for an arbitrary random variable τ satisfying a certain assumption about its short-time distribution. The subsequent corollaries then consider the case that τ is a CTMC FPT as in Sec. II (Corollary 1) and the case that τ is a CTRW FPT as in Sec. III (Corollary 2).

Theorem 5. Let τ be any random variable satisfying

$$\mathbb{P}(\tau \leqslant t_{\min} + t) \sim At^d \quad \text{as } t \to 0+,$$

for some $t_{\min} \geqslant 0, A > 0$, and $d > 0$. Let σ be an independent exponential random variable with rate $\gamma > 0$ and let $m > 0$. If $t_{\min} = 0$, then

$$\mathbb{E}[\tau^m \mid \tau < \sigma] \sim \frac{\Gamma(d + m)}{\Gamma(d)}\frac{1}{\gamma^m} \quad \text{as } \gamma \to \infty. \qquad (49)$$

If $t_{\min} > 0$, then

$$\mathbb{E}[\tau^m \mid \tau < \sigma] - (t_{\min})^m \sim \frac{dm}{\gamma}(t_{\min})^{m-1} \quad \text{as } \gamma \to \infty. \qquad (50)$$

The next two corollaries follow immediately from Theorem 5 and Propositions 1 and 2.

Corollary 1. Let τ be as in Sec. II and let $d \geqslant 1$ be as in Theorem 1. Let σ be an independent exponential random variable with rate γ and let $m > 0$. Equation (49) holds.

Corollary 2. Let τ be as in Sec. III and let $d \geqslant 1$ and $t_{\min} \geqslant 0$ be as in Theorem 3. Let σ be an independent exponential random variable with rate γ and let $m > 0$. If $t_{\min} = 0$, then (49) holds. If $t_{\min} > 0$, then (50) holds.

Theorem 5 and Corollaries 1 and 2 show that FPTs conditioned to be less than a fast inactivation time and extreme FPTs have similar qualitative properties. In particular, compared to unconditioned FPTs, such conditional FPTs are faster, less variable (all the moments vanish), and are less affected by the network size, structure, and details, since they depend only on the minimum number of jumps d that are required to reach the target. Corollary 1 recovers some results proven in Ref. [42] for a discrete state space diffusion model.

VII. DISCUSSION

We have analyzed extreme FPTs for a general class of CTRWs on networks. In the case that there are many searchers (random walkers), we found explicit formulas for the extreme FPT distribution and moments that depend only on the parameters along the geodesic path(s) from the starting location(s) to the target. Hence, the extreme FPTs are independent of the details of the network outside this geodesic(s). We proved similar results for searchers which are conditioned to find the target before a fast inactivation time.

Extreme FPTs have been studied extensively for diffusion processes with continuous state spaces [22–35]. This project was started in 1983 by Weiss, Shuler, and Lindenberg [23], and more recent work has been motivated primarily by biological applications [11]. Interesting work has also been done for extreme FPTs of diffusion on fractals [51,52]. These previous works are marked by an inverse logarithmic decay of the mean extreme FPT,

$$\mathbb{E}[T_N] \propto \frac{1}{\ln N}, \tag{51}$$

as the number of searchers N grows. Indeed, it was recently proven [22] that (51) holds for diffusive search under very general assumptions, as long as the searchers cannot start arbitrarily close to the target. If the diffusive searchers start uniformly in the spatial domain (which means that they can start arbitrarily close to the target), then it was proven in Refs. [53,54] that as $N \to \infty$,

$$\mathbb{E}[T_N] \propto \frac{1}{N^2} \quad \text{or} \quad \mathbb{E}[T_N] \propto \frac{1}{N}, \tag{52}$$

depending on whether the target is perfectly or partially reactive (the result $\mathbb{E}[T_N] \propto N^{-2}$ for a perfectly reactive target was in fact first shown in Ref. [23]).

In contrast to expressions (51) and (52), in the present work we found that extreme FPTs of CTRWs on discrete networks decay as

$$\mathbb{E}[T_N] \propto \frac{1}{N^{1/d}}, \quad \text{as } N \to \infty, \tag{53}$$

where $d \geqslant 1$ is the minimum number of jumps required to reach the target. Comparing (51) and (53), it is clear that the behavior of extreme FPTs of "diffusion" depend critically on whether the diffusion is modeled by a continuous state space or a discrete state space. See Sec. V for more on this discrepancy.

An interesting related work studying extreme FPTs for processes on discrete networks is that of Weng and colleagues [55]. In Ref. [55] the authors investigated the mean of extreme FPTs for *discrete-time* random walks on finite networks (termed the "mean first parallel passage time"). These authors found an exact formula for this mean time in terms of a matrix describing the network structure. Then, upon averaging over starting locations and target locations, they found that this "global" mean time decays as $1/N$ as the number of searchers N grows.

An important line of related works is [6,56–59], which study various network-based measures which generalize the concept of distance. These measures define the "effective distance" between pairs of nodes in a network by taking into account the probabilities of paths between the nodes. Some of this work seeks to understand the arrival time of an infectious disease to a given location. In particular, these works seek to incorporate the idea that frequently traveled routes between two locations (such as airports) make them effectively closer.

In the present work, we similarly found that a certain geodesic path between the source node(s) and the target node(s) controls the extreme FPTs. We found that the geodesic path that is relevant for extreme FPTs minimizes the number of intermediary nodes between the source and target (and the minimum time for the general model in Sec. III). We emphasize that this is a result of the analysis and not an assumption. Indeed, one might have expected that other notions of "optimal" paths [56] or most probable paths would yield the paths taken by the fastest searchers. However, we have found that this is not the case, as the probability of a path or the rates along a path play a strictly secondary role for extreme FPTs.

Another related work is a novel study of the transport efficiency of the endoplasmic reticulum [7], which modeled the endoplasmic reticulum as an active network. These authors found a remarkable mode of transportation, in which molecules group together in seemingly redundant packets at particular locations in the network. Similar to the present work, these authors then found that the extreme FPT out of these many apparently redundant molecules is much faster than a single FPT. The extreme FPTs in Ref. [7] were computed by making a diffusion approximation and applying results for extreme FPTs of diffusion [which yielded an inverse logarithmic decay of the mean extreme FPT as in Eq. (51)].

One final related work is the recent study of Ma and colleagues [42], which considered the effects of a fast inactivation time on mortal diffusive searchers in the context of intracellular signaling. These authors found that if a signaling molecule is conditioned to reach the nucleus before a fast inactivation time, then the FPT is much faster, much less variable, and much less affected by intracellular geometry and obstacles (compared to unconditioned, immortal searchers). As in the case of extreme statistics, such conditioning filters out searchers which deviate from the shortest path to the target. Mathematically, Ref. [42] considered both continuous state space and discrete state space models of diffusive search, and Ref. [43] later considered this problem for continuous state space diffusive search. Corollaries 1 and 2 in Sec. VI extend some results in Ref. [42] to the case of general CTMCs and CTRWs on networks.

APPENDIX

In this Appendix, we prove the propositions and theorems of the main text. We begin with a lemma giving the short-time behavior of the cumulative distribution function of a sum of independent random variables.

Lemma 1. If $\{\sigma_m\}_{m=0}^{d-1}$ are independent random variables with

$$\mathbb{P}(\sigma_m \leqslant t) \sim \lambda_m t^r, \quad \text{as } t \to 0+,$$

for $\lambda_m > 0$ and $r > 0$, then

$$\mathbb{P}\left(\sum_{m=0}^{d-1} \sigma_m \leqslant t\right) \sim \frac{(\Gamma(r+1))^d \prod_{m=0}^{d-1} \lambda_m}{\Gamma(dr+1)} t^{dr}, \text{ as } t \to 0+.$$

Proof of Lemma 1. If X is a random variable with cumulative distribution function $F_X(t) := \mathbb{P}(X \leqslant t)$, then let $\mathcal{L}^*$ denote the Laplace-Stieltjes transform,

$$\mathcal{L}^* F = \{\mathcal{L}^* F\}(s) := \mathbb{E}[e^{-sX}].$$

Letting

$$F_{\sigma_d}(t) := \mathbb{P}(\sigma_d \leqslant t),$$

$$F_{\sum_{m=0}^{d-1} \sigma_d}(t) := \mathbb{P}\left(\sum_{m=0}^{d-1} \sigma_d \leqslant t\right),$$

independence implies that

$$\{\mathcal{L}^* F_{\sum_{m=0}^{d-1} \sigma_d}\}(s) = \mathbb{E}\left[e^{-s\sum_{m=0}^{d-1}\sigma_d}\right] = \prod_{m=0}^{d-1} \mathbb{E}[e^{-s\sigma_d}] \quad \text{(A1)}$$

$$= \prod_{m=0}^{d-1} \{\mathcal{L}^* F_{\sigma_d}\}(s). \quad \text{(A2)}$$

By the Tauberian theorem (see, for example, Theorems 1 and 3 in chapter XIII.5 in Ref. [60]), we have that

$$\{\mathcal{L}^* F_{\sigma_d}\}(s) \sim \Gamma(r+1)\lambda_m s^{-r}, \quad \text{as } s \to \infty.$$

Therefore, (A1) implies that

$$\{\mathcal{L}^* F_{\sum_{m=0}^{d-1}\sigma_d}\}(s) \sim (\Gamma(r+1))^d s^{-dr} \prod_{m=0}^{d-1} \lambda_m, \quad \text{as } s \to \infty.$$

Applying the Tauberian theorem again yields

$$F_{\sum_{m=0}^{d-1}\sigma_d}(t) \sim \frac{(\Gamma(r+1))^d \prod_{m=0}^{d-1} \lambda_m}{\Gamma(dr+1)} t^{dr}, \quad \text{as } t \to 0+,$$

which completes the proof. ∎

Proof of Proposition 1. We first prove the proposition for the case that $\rho(i_0) = 1$ for some fixed $i_0 \in I$ and $\rho(j) = 0$ for all $j \neq i_0$. That is, assume $X(0) = i_0$ almost surely.

Let $M(t) \in \mathbb{N} \cup \{0\}$ be the number of jumps of $X(t)$ before time t. Then

$$\mathbb{P}(\tau \leqslant t) = \mathbb{P}(\tau \leqslant t \cap M(t) \geqslant d), \quad \text{(A3)}$$

since X cannot reach the target from state i_0 unless it makes at least $d = d_{\min}(i_0, I_{\text{target}}) \geqslant 1$ jumps. Since I is countable, there are countably many paths of length d from i_0 to I_{target}. We can thus index the paths so that

$$\mathcal{S}(i_0, I_{\text{target}}) = \{\mathcal{P}_k\}_{k \in K},$$

where $K \subseteq \mathbb{N}$ is some index set. Let $\mathcal{E}_k$ denote the event that X takes path $k \in K$ from i_0 to I_{target}. Notice that

$$\mathbb{P}(\mathcal{E}_k) = \prod_{i=0}^{d-1} \frac{q(\mathcal{P}_k(i), \mathcal{P}_k(i+1))}{q(\mathcal{P}_k(i))} = \frac{\lambda(\mathcal{P}_k)}{\prod_{i=0}^{d} q(\mathcal{P}_k(i))}, \; k \in K. \quad \text{(A4)}$$

Now,

$$\mathbb{P}(\tau \leqslant t \cap M(t) \geqslant d \,|\, \mathcal{E}_k) = \mathbb{P}(M(t) \geqslant d \,|\, \mathcal{E}_k)$$

$$= \mathbb{P}\left[\sum_{m=0}^{d-1} \sigma_m / q(\mathcal{P}_k(m)) < t\right],$$

where $\{\sigma_m\}_{m=0}^{d-1}$ are iid exponential random variables with unit rate. Hence, Lemma 1 and (A4) imply that

$$\mathbb{P}(\tau \leqslant t \cap M(t) \geqslant d \,|\, \mathcal{E}_k)\mathbb{P}(\mathcal{E}_k) \sim \frac{\lambda(\mathcal{P}_k)}{d!} t^d, \quad \text{as } t \to 0+. \quad \text{(A5)}$$

We want to conclude from (A5) that

$$\mathbb{P}(\tau \leqslant t \cap M(t) \geqslant d) = \sum_{k \in K} \mathbb{P}(\tau \leqslant t \cap M(t) \geqslant d \,|\, \mathcal{E}_k)\mathbb{P}(\mathcal{E}_k)$$

$$\sim \sum_{k \in K} \frac{\lambda(\mathcal{P}_k)}{d!} t^d, \quad \text{as } t \to 0+. \quad \text{(A6)}$$

If $|K| < \infty$, then this is immediate. To handle the case that $|K| = \infty$, notice that Lemma 1 implies that

$$\mathbb{P}(\tau \leqslant t \cap M(t) \geqslant d \,|\, \mathcal{E}_k)$$

$$= \mathbb{P}\left[\sum_{m=0}^{d-1} \sigma_m / q(\mathcal{P}_k(m)) < t\right]$$

$$\leqslant \mathbb{P}\left(\sum_{m=0}^{d-1} \sigma_m / \overline{q} < t\right) \sim \frac{\overline{q}^d}{d!} t^d, \quad \text{as } t \to 0+,$$

where $\overline{q} := \sup_i q(i) < \infty$. Therefore, there exists an $\varepsilon > 0$ that is independent of $k \in K$ so that

$$\frac{\mathbb{P}(\tau \leqslant t \cap M(t) \geqslant d \,|\, \mathcal{E}_k)\mathbb{P}(\mathcal{E}_k)}{t^d}$$

$$\leqslant 2\frac{\overline{q}^d}{d!}\mathbb{P}(\mathcal{E}_k), \quad \text{for all } t \in (0, \varepsilon), \; k \in K. \quad \text{(A7)}$$

Since

$$\sum_{k \in K} 2\frac{\overline{q}^d}{d!}\mathbb{P}(\mathcal{E}_k) = 2\frac{\overline{q}^d}{d!} < \infty, \quad \text{(A8)}$$

Lebesgue's dominated convergence theorem yields (A6), which then completes the proof for the case $\rho(i_0) = 1$ due to (A3).

To handle the case of a general initial distribution ρ on I, observe that

$$\mathbb{P}(\tau \leqslant t) = \sum_{i \in \operatorname{supp}(\rho)} \mathbb{P}(\tau \leqslant t \,|\, X(0) = i)\rho(i). \qquad (A9)$$

The desired result is then immediate if the support of ρ is finite. A similar application of the dominated convergence theorem as above completes the proof for the case that I is infinite. In particular, as in (A7) and (A8) we have that if $t \in (0, \varepsilon)$, then

$$
\begin{aligned}
t^{-d}\mathbb{P}(\tau \leqslant t \,|\, X(0) = i)\rho(i) \\
= \rho(i)\sum_{k \in K} t^{-d}\mathbb{P}(\tau \leqslant t \,|\, X(0) = i \cap \mathcal{E}_k)\mathbb{P}(\mathcal{E}_k) \\
\leqslant \rho(i)2\frac{\overline{q}^d}{d!}.
\end{aligned}
$$

Since $\sum_{i \in \operatorname{supp}(\rho)} \rho(i)2\overline{q}^d/d! = 2\overline{q}^d/d! < \infty$, the proof is complete. ∎

Proof of Theorem 1. The result follows directly from Proposition 1 and Theorems 2 and 3 in Ref. [53]. ∎

Proof of Theorem 2. The result follows directly from Proposition 1 and Theorems 5 and 6 in Ref. [53]. ∎

Proof of Proposition 2. As in the proof of Proposition 1, we first consider the case that $\rho(i_0) = 1$ for some fixed $i_0 \in I$ and $\rho(j) = 0$ for all $j \neq i_0$. That is, we assume $X(0) = i_0$ almost surely.

Since I is finite, it is immediate that there exists $\varepsilon > 0$ so that $t_0(\mathcal{P}) > t_{\min} + \varepsilon$ for all $\mathcal{P}$ with $t_0(\mathcal{P}) \neq t_{\min}(i_0, I_{\text{target}})$. Hence, if $\mathcal{E}$ denotes the event that X takes a path $\mathcal{P}$ with $t_0(\mathcal{P}) \neq t_{\min}(i_0, I_{\text{target}})$, then

$$\mathbb{P}(\tau \leqslant t_{\min} + t \cap \mathcal{E}) = 0, \quad \text{for all } t < \varepsilon.$$

Therefore, if we index all the paths $\mathcal{P}$ with $t_0(\mathcal{P}) = t_{\min}(i_0, I_{\text{target}})$ as $\{\mathcal{P}_k\}_{k \in K}$ and let $\mathcal{E}_k$ denote the event that X takes path $\mathcal{P}_k$, then

$$\mathbb{P}(\tau \leqslant t_{\min} + t) \sim \sum_{k \in K} \mathbb{P}(\tau \leqslant t_{\min} + t \cap \mathcal{E}_k), \quad \text{as } t \to 0+.$$

Let $M(t) \in \mathbb{N} \cup \{0\}$ be the number of jumps of $X(t)$ before time t. Now,

$$
\begin{aligned}
\mathbb{P}(\tau &\leqslant t_{\min} + t \cap \mathcal{E}_k) \\
&= \mathbb{P}(\tau \leqslant t_{\min} + t \cap M(t_{\min} + t) \geqslant d \cap \mathcal{E}_k), \quad k \in K,
\end{aligned}
$$

since if X takes path $\mathcal{P}_k$ with $k \in K$, then it must make at least $d = d_{\min}(i_0, I_{\text{target}}) \geqslant 1$ jumps to reach the target.

Next, let $K' \subseteq K$ be an index set so that $\{\mathcal{P}_k\}_{k \in K'}$ are the set of paths in $\mathcal{S}(i_0, I_{\text{target}})$. It is then immediate that

$$
\begin{aligned}
\mathbb{P}(\tau \leqslant t_{\min} + t \cap M(t_{\min} + t) &\geqslant d \,|\, \mathcal{E}_k) \\
&= \mathbb{P}(M(t_{\min} + t) \geqslant d \,|\, \mathcal{E}_k), \quad k \in K'.
\end{aligned}
$$

Further, if X takes path $\mathcal{P}_k$ with $k \in K'$ and $M(t_{\min} + t) \geqslant d$, then

$$\sum_{m=0}^{d-1} \sigma_m \leqslant t_{\min} + t,$$

where $\{\sigma_m\}_{m=0}^{d-1}$ are the $d \geqslant 1$ waiting times. In particular, σ_m has the distribution

$$\mathbb{P}(\sigma_m \leqslant s) = F_{\mathcal{P}_k(m), \mathcal{P}_k(m+1)}(s).$$

Therefore, if we define

$$\widetilde{\sigma}_m = \sigma_m - t_0(\mathcal{P}_k(m), \mathcal{P}_k(m+1)),$$

then $\mathbb{P}(\widetilde{\sigma}_m \leqslant s) = \widetilde{F}_{\mathcal{P}_k(m), \mathcal{P}_k(m+1)}(s)$, where

$$
\begin{aligned}
\widetilde{F}&_{\mathcal{P}_k(m), \mathcal{P}_k(m+1)}(s) \\
&:= F_{\mathcal{P}_k(m), \mathcal{P}_k(m+1)}[s + t_0(\mathcal{P}_k(m), \mathcal{P}_k(m+1))],
\end{aligned}
$$

and thus as $s \to 0+$,

$$\widetilde{F}_{\mathcal{P}_k(m), \mathcal{P}_k(m+1)}(s) \sim \lambda(\mathcal{P}_k(m), \mathcal{P}_k(m+1))s^r.$$

Therefore, by Lemma 1, we have that for $k \in K'$,

$$
\begin{aligned}
\mathbb{P}(\tau &\leqslant t_{\min} + t \cap M(t_{\min} + t) \geqslant d \,|\, \mathcal{E}_k) \\
&= \mathbb{P}\left[\sum_{m=0}^{d-1} \sigma_m \leqslant t_{\min} + t\right] \\
&= \mathbb{P}\left\{\sum_{m=0}^{d-1} \left[t_0(\mathcal{P}_k(m), \mathcal{P}_k(m+1)) + \widetilde{\sigma}_m\right] \leqslant t_{\min} + t\right\} \\
&= \mathbb{P}\left(\sum_{m=0}^{d-1} \widetilde{\sigma}_m \leqslant t\right) \sim \frac{(\Gamma(r+1))^d \lambda(\mathcal{P}_k)}{\Gamma(dr+1)} t^{dr}, \quad \text{as } t \to 0+.
\end{aligned}
$$

Noting that $\mathbb{P}(\mathcal{E}_k) = \pi(\mathcal{P}_k)$ and summing over k completes the proof for the case $\rho(i_0) = 1$ (note that $|K| < \infty$ since $|I| < \infty$). The case of a general distribution ρ on I is handled analogously to (A9). ∎

Proof of Theorem 3. The result follows directly from Proposition 2 and Theorems 2 and 3 in Ref. [53]. ∎

Proof of Theorem 4. The result follows directly from Proposition 2 and Theorems 5 and 6 in Ref. [53]. ∎

Proof of Theorem 5. Lemma 3 in Ref. [43] gives the following representation for the conditional mth moment,

$$
\begin{aligned}
\mathbb{E}[\tau^m \,|\, \tau < \sigma] &= \frac{\int_0^\infty t^m \gamma e^{-\gamma t} F(t)\, dt}{\int_0^\infty \gamma e^{-\gamma t} F(t)\, dt} \\
&\quad - \frac{\int_0^\infty m t^{m-1} e^{-\gamma t} F(t)\, dt}{\int_0^\infty \gamma e^{-\gamma t} F(t)\, dt}, \quad (A10)
\end{aligned}
$$

where $F(t) := \mathbb{P}(\tau \leqslant t)$. First, suppose that $t_{\min} = 0$. Let $\varepsilon \in (0, 1)$. By assumption, there exists $\delta > 0$ so that

$$(1 - \varepsilon)At^d < F(t) < (1 + \varepsilon)At^d, \quad \text{for all } t \in (0, \delta).$$

Therefore, for any $n > -1$, we have that

$$
\begin{aligned}
(1 - \varepsilon)\int_0^\delta t^{n+d} e^{-\gamma t}\, dt &< \frac{1}{A}\int_0^\delta t^n e^{-\gamma t} F(t)\, dt \\
&< (1 + \varepsilon)\int_0^\delta t^{n+d} e^{-\gamma t}\, dt. \quad (A11)
\end{aligned}
$$

Now, it is a straightforward to check that

$$\int_0^\delta t^{n+d} e^{-\gamma t}\, dt \sim \int_0^\infty t^{n+d} e^{-\gamma t}\, dt, \quad \text{as } \gamma \to \infty,$$

since $\int_\delta^\infty t^{n+d} e^{-\gamma t} \, dt$ vanishes exponentially as $\gamma \to \infty$. Furthermore, it is a simple calculus exercise to check that

$$\int_0^\infty t^{n+d} e^{-\gamma t} \, dt = \frac{\Gamma(d+n+1)}{\gamma^{d+n+1}}.$$

Since $\int_\delta^\infty t^n e^{-\gamma t} F(t) \, dt$ vanishes exponentially as $\gamma \to \infty$ and since $\varepsilon > 0$ is arbitrary in (A11), we thus obtain

$$\int_0^\infty t^n e^{-\gamma t} F(t) \, dt \sim \frac{A\Gamma(d+n+1)}{\gamma^{d+n+1}}, \quad \text{as } \gamma \to \infty. \quad \text{(A12)}$$

Combining (A12) with (A10) and simplifying yields (49).

Next, suppose $t_{\min} > 0$. As above, it is straightforward to check that if $n > -1$, then as $\gamma \to \infty$ we have

$$\int_0^\infty t^n e^{-\gamma t} F(t) \, dt \sim \int_{t_{\min}}^\infty t^n e^{-\gamma t} A(t - t_{\min})^d \, dt. \quad \text{(A13)}$$

Furthermore, changing variables yields

$$\int_{t_{\min}}^\infty e^{-\gamma t} (t - t_{\min})^d \, dt = e^{-\gamma t_{\min}} \frac{\Gamma(d+1)}{\gamma^{d+1}}. \quad \text{(A14)}$$

In addition, a straightforward application of Watson's lemma gives

$$\left[\int_{t_{\min}}^\infty t^m e^{-\gamma t} (t - t_{\min})^d \, dt - e^{-\gamma t_{\min}} \frac{\Gamma(d+1)(t_{\min})^m}{\gamma^{d+1}} \right]$$
$$\sim e^{-\gamma t_{\min}} \frac{m\Gamma(d+2)(t_{\min})^{m-1}}{\gamma^{d+2}}, \quad \text{as } \gamma \to \infty. \quad \text{(A15)}$$

Similarly, Watson's lemma also gives that as $\gamma \to \infty$,

$$\int_{t_{\min}}^\infty t^{m-1} e^{-\gamma t} (t - t_{\min})^d \, dt \sim e^{-\gamma t_{\min}} \frac{\Gamma(d+1)(t_{\min})^{m-1}}{\gamma^{d+1}}. \quad \text{(A16)}$$

Combining (A13)–(A16) with (A10) completes the proof. ∎

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