Schur-Weyl duality

These notes are from a graduate student reading group at the University of Utah happening during the Fall semester 2018.

Contents

1 Introduction **2**

2 Modules and representations of groups **3**
2.1 Modules . . . 3
2.2 Group representations . . . 4

3 Linear algebra **6**
3.1 Tensor products . . . 6
3.1.1 Application to representations . . . 8
3.2 Symmetric products . . . 10
3.3 Exterior products . . . 12
3.4 Determinant . . . 13

4 Character theory **13**
4.1 Decomposition of Representations . . . 14
4.2 Character Theory . . . 15
4.2.1 Properties of χ . . . 15
4.2.2 More Properties of χ . . . 16

5 Representations of the symmetric group **16**

6 Double Centralizer Theorem **24**

7 Schur-Weyl duality **27**

8 Determinental rings and applications to commutative algebra **29**
8.1 Determinantal Rings Are Cohen-Macaulay . . . 30
8.2 Relation to Representation Theory . . . 31

9 Decomposing tensor products of Weyl modules **33**

1 Introduction

Talk by Adam Brown, notes by Sabine Lang

We give here a brief outline of the topics that will be covered during the semester, along with some motivations behind Schur-Weyl duality. The basic idea comes from Lie groups: we want to find all the representations of these Lie groups. In our case, we will focus on only one Lie group: the general linear group.

For $V = \mathbb{C}^n$, we define $G = \mathbf{GL}(V)$ and want to find all the representations of G, i.e., the continuous homomorphisms $\pi : G \to GL(W)$ for W a complex vector space. We also say that G acts on W in this case. As a starting point, we know one representation: the canonical or standard representation, defined by $\mathrm{Id} : G \to \mathbf{GL}(V)$. Can we use this to construct new representations?

Let us first analyze the operations that we can do with vector spaces. The direct sum of V with itself does not give a new representation: $G \to GL(V \oplus V)$ is given by $g \to \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$ which corresponds to two copies of the standard representation. A more interesting operation is the tensor product: G acts on $V \otimes V$ by $g \cdot (v_1 \otimes v_2) = (g \cdot v_1) \otimes (g \cdot v_2)$. How can we decompose $V \otimes V$ into G-invariant subspaces?

A first subspace which is preserved by the action of G is the space of alternating vectors $\Lambda^2 V = \langle \{v_1 \otimes v_2 \mid v_1 \otimes v_2 = -v_2 \otimes v_1\} \rangle$. Can we decompose $V \otimes V$ as $\Lambda^2 V \oplus W$? Yes! And W is then equal to $\mathrm{Sym}^2 V = \langle \{v_1 \otimes v_2 \mid v_1 \otimes v_2 = v_2 \otimes v_1\} \rangle$ and is also preserved by the action of G. We obtain $V \otimes V = \Lambda^2 V \oplus \mathrm{Sym}^2 V$.

Can we generalize this idea? We can consider $\overbrace{V \otimes \cdots \otimes V}^{k} = V^{\otimes k}$. However, the decomposition that we had for $k = 2$ is no longer as simple. For $k = 3$ already, $V^{\otimes 3} = \Lambda^3 V \oplus \mathrm{Sym}^3 V \oplus W$ with $W \neq 0$. Therefore, one of our goals is to decompose $V^{\otimes k}$ into G-invariant subspaces.

Let us consider the symmetric group $S_k = \{$ bijections from F_k to $F_k\}$, where F_k is a set with k elements. We have an action of S_k on F_k by $\sigma \cdot x = \sigma(x)$ for $\sigma \in S_k, x \in F_k$. However, F_k is a set and not a vector space, so this is not a representation. But we can construct a vector space $E = \{f : F_k \to \mathbb{C}\}$, which has an action of S_k given by $(\sigma \cdot f)(x) = f(\sigma^{-1}(x))$. This is a representation of S_k.

Going back to $V^{\otimes k}$, we can use it to construct a representation of S_k: we have an action of the symmetric group on $V^{\otimes k}$ given by $\sigma \cdot (v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$ on simple tensors, and extended linearly. Now, we can try to decompose $V^{\otimes k}$ into S_k-invariant subspaces.

For example, when $k = 2$, the group S_2 has two elements, $\begin{pmatrix}1 & 2\end{pmatrix}$ and the identity. If $v_1 \otimes v_2 \in \Lambda^2 V$, then $\begin{pmatrix}1 & 2\end{pmatrix} \cdot v_1 \otimes v_2 = v_2 \otimes v_1 = -v_1 \otimes v_2$. Therefore, S_2 acts on $\Lambda^2 V$ by the sign of the permutation, and we can decompose $\Lambda^2 V$ into copies of the sign representation. Now if $v_1 \otimes v_2 \in \mathrm{Sym}^2 V$, then $\begin{pmatrix}1 & 2\end{pmatrix} \cdot v_1 \otimes v_2 = v_2 \otimes v_1 = v_1 \otimes v_2$, and $\mathrm{Sym}^2 V$ decomposes into copies of the trivial representation. We conclude that $V \otimes V = \Lambda^2 V \oplus \mathrm{Sym}^2 V$ is a decomposition into invariant subspaces, both for the G-action and for the S_2-action.

This turns out to be true in general. This is due to the fact (to be proven later) that the actions of S_k and G commute: for $\sigma \in S_k$, $g \in G$ we have $\sigma \cdot (g \cdot (v_1 \otimes \cdots \otimes v_k)) = g \cdot (\sigma \cdot (v_1 \otimes \cdots \otimes v_k))$. Therefore, the decompositions into invariant subspaces should agree. This leads to the main theorem of Schur-Weyl duality:

Theorem (Schur-Weyl duality)**.** *$V^{\otimes k}$ decomposes both as $\oplus_{\pi \in Irr(S_k)} E_\pi$ and as $\oplus_{\phi \in Irr(\mathbf{GL}(V))} U_\phi$.*

We are in particular interested in using S_k (it is a finite group, and there is a combinatorial approach to its representations) to understand $\mathbf{GL}(V)$. In conclusion, if we have a canonical representation V of $G = \mathbf{GL}(V)$ and an irreducible representation π of S_k, then we can construct an irreducible representation of G. More precisely, we can get all polynomial representations of G that way. We can realize this using the Schur functor

$$\mathbb{S}_\pi : V \mapsto \mathrm{Hom}_{S_k}(E_\pi, V^{\otimes k}),$$

and $\mathrm{Hom}_{S_k}(E_\pi, V^{\otimes k})$ is an irreducible representation of G when $\dim(V) = k$ or 0.

This duality has applications to any object with both a $\mathbf{GL}(V)$ and an S_k action. For example, for a field k, we can define $k[\begin{pmatrix} x & y \\ w & z \end{pmatrix}]$ (think of $k[x]$, but we add a matrix instead of a variable x). Then $k[\begin{pmatrix} x & y \\ w & z \end{pmatrix}]/(xz - yw)$ has a natural $\mathbf{GL}(V)$-action. We can use Schur-Weyl duality to decompose it, and compute the syzygies.

2 Modules and representations of groups

Talk by Cameron Zhao, notes by Peter McDonald

Today we are going to introduce some fundamental tools of linear algebra that we will be using later in the semester.

2.1 Modules

Definition 2.1.1. *Let R be an associative algebra with unity. An R-module is an abelian group M with an action of R on M satisfying*

1. *The action preserves addition and multiplication in R and addition in M, i.e., $(rs + t) \cdot (a + b) = r(s \cdot a) + t \cdot a + r \cdot (s \cdot b) + t \cdot b$ for $r, s, t \in R$ and $a, b \in M$*
2. *$1_R \cdot a = a$ for all $a \in M$*

Example 2.1.2. *Abelian groups are $\mathbb{Z}$-modules*

Example 2.1.3. *Vector spaces over a field k are k-modules.*

Example 2.1.4. *Consider the vector space k^n for a field k. This is a module over the matrix algebra $M_n(k)$.*

Example 2.1.5. *Take $A \in M_n(k)$. Then k^n can be viewed as a $k[x]$-module where x acts as A. This gives us a lot of information about A: rational and Jordan canonical forms, minimal polynomial, etc.*

Example 2.1.6. *If R is an algebra, then R is a left R-module. The submodules of R are precisely the left ideals of R.*

Example 2.1.7. *Let R be a commutative ring and M a left R-module. Then M is also a right R-module where the action is given by $a \cdot r = r \cdot a$ for $r \in R$ and $a \in M$. Then M is a bi-module.*

Remark 2.1.8. *If R is not commutative then the above is not well-defined because*

$$a \cdot (rs) = (a \cdot r) \cdot s = (r \cdot a) \cdot s = s \cdot (r \cdot a) = (sr) \cdot a \neq (rs) \cdot a$$

2.2 Group representations

Definition 2.2.1. *Let G be a group. A representation of G is a group homomorphism $p : G \to \mathbf{GL}(V)$ for some vector space V.*

Group representations are precisely modules over a certain algebra, so we can use the tools to study modules to study representations. What is this algebra though?

Definition 2.2.2. *The group algebra of G over a field k, denoted $k[G]$, is constructed as the k-vector space spanned by the basis G, i.e., $k(G) = span_k(G)$. Then, every element looks like*

$$\sum_{g \in G} c_g \cdot [g].$$

Multiplication is defined on basis elements by

$$[g_1][g_2] = [g_1 g_2],$$

and extended linearly.

Example 2.2.3. $\mathbb{C}[\mathbb{Z}/3] = \{c_0[0] + c_1[1] + c_2[2] : c_i \in \mathbb{C}\}$.

$$\begin{aligned}
(2[0] + [1])(i\sqrt{2}[1] + 3[2]) &= 2i\sqrt{2}[0][1] + 6[0][2] + i\sqrt{2}[1][1] + 3[1][2] \\
&= 2i\sqrt{2}[0+1] + 6[0+2] + i\sqrt{2}[1+1] + 3[1+2] \\
&= 3[0] + 2i\sqrt{2}[1] + (6 + i\sqrt{2})[2]
\end{aligned}$$

We can show this is an associative algebra with identity because the identity is $1_k[e]$ where $e \in G$ is the identity element of the group.

Proposition 2.2.4. *Representations of the group G over the field k are $k[G]$-modules.*

Proof. If we have a representation $\rho : G \to \mathbf{GL}(V)$ then it extends linearly to $\tilde{\rho} : k[G] \to \mathrm{End}_k(V)$. Then V is a $k[G]$-module.

Consider the following example of how $k[G]$ acts on V. Consider how $c_1g_1 + c_2g_2 \in k[G]$ acts on $v \in V$:

$$\begin{aligned}(c_1g_1 + c_2g_2) \cdot v &= \tilde{\rho}(c_1g_1 + c_2g_2) \cdot v \\ &= c_1\rho(g_1) \cdot v + c_2\rho(g_2) \cdot v \\ &= \tilde{\rho}(c_1g_2) \cdot v + \tilde{\rho}(c_2g_2) \cdot v \\ &= (c_2g_2) \cdot v + (c_2g_2) \cdot v\end{aligned}$$

Now, given a $k[G]$-module V, we get a ring homomorphism $\tilde{\rho} : k[G] \to \mathrm{End}_k(V)$ where $\tilde{\rho}$ maps groups elements of G to invertible transformations:

$$\tilde{\rho}([g])\tilde{\rho}([g^{-1}]) = \tilde{\rho}([g][g^{-1}]) = \tilde{\rho}(e) = \mathrm{Id}.$$

Because $\tilde{\rho}$ preserves multiplication, $\rho = \tilde{\rho}|_G$ is a group homomorphism. Then $\rho : G \to \mathbf{GL}(V)$ is a representation. □

Essentially, instead of thinking of G acting on a set, we are thinking about G as giving us functions on our set. Now we need to make sure that $k[G]$-homomorphisms are the same as homomorphisms on representations.

Definition 2.2.5. *A homomorphism of representations $\varphi : V \to W$ is a k-linear map that commutes with the group action, i.e., the following diagram commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow \\ V & \xrightarrow{\varphi} & W \end{array}$$

This is also called a G-equivariant map.

Definition 2.2.6. *An R-module homomorphism $\varphi : M \to N$ is a $\mathbb{Z}$-linear map that commutes with the ring action, i.e., $r \cdot \varphi(m) = \varphi(r \cdot m)$ for $r \in R$ and $m \in M$. This is also called an R-linear map.*

So a $k[G]$-linear map $\varphi : V \to W$ is one such that for all $\sum_{g \in G} c_g[g] \in k[G]$ the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle \sum_{g\in G} c_g[g]}\downarrow & & {\scriptstyle \sum_{g\in G} c_g[g]}\downarrow \\ V & \xrightarrow{\varphi} & W \end{array}$$

Take $1[e] = [e]$, then considering the following diagram we can see that φ is k-linear:

$$\begin{CD} V @>{\varphi}>> W \\ @V{[e]}VV @V{[e]}VV \\ V @>{\varphi}>> W \end{CD}$$

Furthermore, if you take $[g]$, then φ becomes a homomorphism between representations, so a $k[G]$-module homomorphism is also a homomorphism of representations:

$$\begin{CD} V @>{\varphi}>> W \\ @V{g}VV @V{g}VV \\ V @>{\varphi}>> W \end{CD}$$

3 Linear algebra

Talk by Cameron Zhao, notes by Cameron Zhao

(Some of the contents in this section were skipped in the talk).

3.1 Tensor products

Recall from last time: we showed that $k[G]$-modules are the same as representations of G, and that $k[G]$-module maps are the same as homomorphisms between representations. In other words, the category of $k[G]$-modules is isomorphic to the category of representations of G.

The direct product $M \times N$ of two modules is again a module, where $r(m,n) = (rm, rn)$. It is also called the **direct sum**, denoted $M \oplus N$ (note: direct sums and direct products of modules only differ when infinitely many modules are considered). The idea is that when we take the direct sum of two vector spaces, the resulting dimension is the sum of the dimension of the two spaces. So one can try to construct a vector space whose dimension is the product of two smaller spaces. A natural choice of basis on such a space is $\{(e_i, f_j)\}$. This is the tensor product.

This can be done for general modules. It should be an abelian group satisfying some conditions. So to do this, we take elements in the free abelian group generated by $M \times N$ and impose the relations we want:

Definition 3.1.1. *Let M be a right R-module and N a left R-module. The tensor product is defined to be $M \otimes_R N := F(M \times N)/I$, where $F(M \times N)$ is the free abelian group generated by $M \times N$, and I is the ideal generated by elements of the form*

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n),$$

$$(m, n_1 + n_2) - (m, n_1) - (m, n_2),$$
$$(mr, n) = (m, rn).$$

The image of (m, n) in the quotient is denoted by $m \otimes n$. So $M \otimes_R N$ consists of elements of the form $\sum_i m_i \otimes n_i$ subject to the relations

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n,$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2,$$

$$mr \otimes n = m \otimes rn.$$

An element of the form $m \otimes n$ is called a ***simple tensor*** *or a* ***rank one tensor****. So $M \otimes_R N$ consists of R-linear combinations of simple tensors.*

The most important property of the tensor product is its universal property. Many characterizations of the tensor product can be deduced from it.

Definition 3.1.2. *A map $\varphi \colon M \times N \to L$ is called R-balanced if it is $\mathbb{Z}$-linear in both arguments and $\varphi(mr, n) = \varphi(m, rn)$.*

Theorem 3.1.3 (Universal Property of Tensor Product)**.** *Let R be an algebra with 1, M a right module, N a left module, and L an abelian group. Then we have a 1-1 correspondence correspondence*

$$\{R\text{-balanced maps } M \times N \xrightarrow{\varphi} L\} \longleftrightarrow \{\text{group homomorphisms } M \otimes_R N \xrightarrow{\phi} L\}$$
$$\varphi \mapsto \phi, \text{ where } \phi(m \otimes n) = \varphi(m, n)$$

such that the following diagram commute:

$$\begin{array}{ccc} M \times N & \xrightarrow{\iota} & M \otimes_R N \\ & \searrow^{\varphi} & \downarrow \phi \\ & & L \end{array}$$

where $\iota(m, n) = m \otimes n$.

Proof. Given ϕ, let $\varphi = \phi \circ \iota$, then φ is R-balanced because $\iota(mr, n) = mr \otimes n = m \otimes rn = \iota(m, rn)$. Given φ, it extends to a group homomorphism $\tilde{\varphi} \colon F(M \times N) \to L$ from the free abelian group $F(M \times N)$. We want it to factor through $M \otimes_R N$, so that it gives a map $\phi : M \otimes_R N \to L$. Indeed, the generators of the ideal defining the tensor product vanish under $\tilde{\varphi}$. □

The universal property can be understood in the following way: we want to study bilinear maps, but bilinear maps are not module maps. The universal property states that using the tensor product we can encodes all information in the bilinear map in a module map.

A commutative version of this is: bilinear maps $M \times N \to L$ are the same as linear maps $M \otimes_R N \to L$.

Using this we can prove many nice properties of the tensor product:

Proposition 3.1.4.

1. *(Extension of scalars) If M is a left R-module and S is an R-algebra, then $S \otimes_R M$ is the "smallest" S-module containing M.*

2. *(Tensor products are associative) $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$ for any left module M, bimodule N and right module L.*

3. *(Tensor products are commutative) If R is commutative, then $M \otimes_R N \cong N \otimes_R M$.*

4. *(Tensor product distributes with direct sums) $(M \oplus N) \otimes_R L \cong (M \otimes_R L) \oplus (N \otimes_R L)$.*

5. *(Vector spaces) If k is a field and V, W are vector spaces with bases $\{e_i\}, \{f_j\}$ respectively, then $V \otimes_k W$ is a vector space with basis $\{e_i \otimes f_j\}$, so $\dim(V \otimes W) = (\dim V) \cdot (\dim W)$.*

Intuitively, when tensoring over fields, the product is not "shrunk" by relations. But over general algebras the product will "shrink". For example:

Example 3.1.5. *Let R be a commutative ring, $I \subset R$ an ideal, M a module. Then $R/I \otimes_R M \cong M/IM$. So if J is another ideal, then $R/I \otimes_R R/J \cong (R/J)/(I/J) \cong R/(I+J)$.*

If we take a bimodule M we can tensor it with itself over and over again. Since tensor products are associative, we have formed the tensor powers $M^{\otimes n}$. We can sum up all the tensor powers and form the **tensor algebra**

$$T(M) = R \oplus M \oplus M^{\otimes 2} \oplus \cdots$$

where the multiplication is given by concatenation of tensors. This is an associative algebra with 1, and is the "largest" associative algebra containing M.
Note: Countable direct sum is defined to be

$$\bigoplus_{i=1}^{\infty} M_i := \{(m_1, m_2, \ldots) \mid m_i \in M_i, \text{ only finitely many entries are nonzero}\},$$

whereas countable direct product is

$$\prod_{i=1}^{\infty} M_i := \{(m_1, m_2, \ldots) \mid m_i \in M_i\},$$

without the finiteness assumption.

3.1.1 Application to representations

If $A : M_1 \to M_2, B : N_1 \to N_2$ are linear maps, then we can construct the tensor product $A \otimes B : M_1 \otimes N_1 \to M_2 \otimes N_2, m \otimes n \mapsto A(m) \otimes B(n)$. So if A', B' are also linear maps, then $(A \otimes B) \circ (A' \otimes B) = (A \circ A') \otimes (B \circ B')$. Over fields and finite dimensional vector spaces, A, B can be represented as matrices. The matrix of $A \otimes B$ is

$$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1l}B \\ \vdots & \vdots & & \vdots \\ a_{s1}B & a_{s2}B & \cdots & a_{sl}B \end{bmatrix},$$

assuming $A = (a_{ij})_{i=1,\dots,s, j=1,\dots,l}$. In particular if A is $s \times s$, B is $l \times l$, then $\operatorname{tr}(A \otimes B) = \operatorname{tr} A \operatorname{tr} B$, $\det(A \otimes B) = (\det A)^s (\det B)^l$.

If $\rho : G \to \mathbf{GL}(V)$, $\sigma : G \to \mathbf{GL}(W)$ are two representations, then for any $g \in G$, $\rho(g) \otimes \sigma(g)$ is an invertible transformation on $V \otimes_k W$. Let $(\rho \otimes \sigma)(g) = \rho(g) \otimes \sigma(g)$, then $\rho \otimes \sigma : G \to \mathbf{GL}(V \otimes_k W)$ is another representation. We will simply write $V \otimes W$.

(**Note**: $V \otimes_k W$ is different from $V \otimes_{k[G]} W$! For example if $V = V_{trv}$ is the trivial representation, then $V_{trv} \otimes_k W \cong W$, but $V_{trv} \otimes_{k[G]} W \cong W_G := W/\langle gw - w \mid g \in G \rangle$.)

Similarly, if $\sigma : H \to \mathbf{GL}(W)$ is a representation of another group, then $\rho \otimes \sigma : G \times H \to \mathbf{GL}(V \otimes_k W)$ is a representation of $G \times H$. This is denoted by $V \boxtimes W$.

Now we look at Hom sets. Let $N^* := \operatorname{Hom}_R(N, R)$ be the dual space of N.

Proposition 3.1.6. *Let M be a free module, i.e., $M \cong R^{\oplus m}$ for some m. There is a $\mathbb{Z}$-linear isomorphism $N \otimes_R M^* \to \operatorname{Hom}_R(M, N), (n \otimes f) \mapsto (m \mapsto nf(m))$. If R is commutative, then it is also R-linear.*

Proof. For simplicity, we prove it for the case when $R = k$ is a field. Any linear map $A : V \to W$ is uniquely determined by its value on basis elements. If $\operatorname{span}\{e_i\} = V$, $\operatorname{span}\{f_j\} = W$, then A is uniquely determined by the coefficient c_{ij}'s where $Ae_i = \sum_j c_{ij} f_j$. So A is the image of $\sum_{ij} c_{ij} f_j \otimes e_i^*$. □

This is very useful. Recall that any linear map $A : V \to W$ induces a map $A^* : W^* \to V^*$, $f \mapsto f \circ A$. If V, W are finite dimensional, then the matrix of A^* is the transpose of the matrix of A. If $\rho : G \to \mathbf{GL}(V)$, $\sigma : G \to \mathbf{GL}(W)$ are representations, then $\rho^* : G \to \mathbf{GL}(V^*)$, $g \mapsto - \circ \rho(g^{-1})$ is a representation. Therefore $\operatorname{Hom}_k(V, W) = W \otimes_k V^*$ is also a representation. The link of this to characters is that $\dim_k(\operatorname{Hom}_k(V, W)^G) = \langle \chi_V, \chi_W \rangle$, where $\operatorname{Hom}_k(V, W)^G$ is the invariant subspace of $\operatorname{Hom}_k(V, W)$ consisting of the elements that are fixed under all $g \in G$.

We mentioned that extension of scalars can be done using tensor products. If $H \leq G$ is a subgroup, then $k[H] \subset k[G]$ is a subalgebra. So if V is an H-module, then we can extend the scalars to $k[G]$ by taking the tensor product.

Definition 3.1.7. *Let $H \leq G$ be a subgroup, and V be an H-module. The **induced representation** from H to G obtained from V is defined to be $\operatorname{Ind}_H^G V := k[G] \otimes_{k[H]} V$. The **coinduced representation** from H to G obtained from V is $\operatorname{Coind}_H^G V := \operatorname{Hom}_H(k[G], V)$.*

This definition is much cleaner than the one that does not use tensor product. An important fact about induction is the Frobenius reciprocity, which is a direct collorary of the following general fact:

Proposition 3.1.8 (Adjoint pairs)**.** *Given a right R-module M, an (R,S)-bimodule N and a right S-module L, there is a unique abelian group isomorphism*

$$\eta\colon \operatorname{Hom}_S(M\otimes_R N, L) \xrightarrow{\sim} \operatorname{Hom}_R(M, \operatorname{Hom}_S(N,L)),$$

given by

$$\eta(f)(m)(n) = f(m\otimes n), \text{ for } f\in \operatorname{Hom}_S(M\otimes_R N, L), m\in M, n\in N.$$

Pictorially

$$\begin{array}{ccc} M & & M\otimes_R N \\ \downarrow{\scriptstyle \eta(f)} & & \downarrow{\scriptstyle f} \\ \operatorname{Hom}_S(N,L) & & L \end{array}$$

As a result,

Corollary 3.1.9. *If $H\le G$ is a subgroup, W is a G-module, V is an H-module, then*

$$\operatorname{Hom}_H(V, \operatorname{Res}^G_H W) \cong \operatorname{Hom}_G(\operatorname{Ind}^G_H V, W).$$

Proof. We only need to show that $\operatorname{Res}^G_H W = \operatorname{Hom}_G(k[G], W)$. In general, if M is an R-module, then $\operatorname{Hom}_R(R,M) \xrightarrow{\sim} M, f\mapsto f(1)$. So $\operatorname{Hom}_G(k[G],W)\cong W$ as vector spaces. The action of H on $k[G]$ makes $\operatorname{Hom}_G(k[G],W)$ an H-module. □

3.2 Symmetric products

In general $m\otimes m' \neq m'\otimes m$ in $M\otimes_R M$. But we can make them equal by taking a quotient.

Definition 3.2.1. *Let R be commutative, and let M be an R-module. The* ***symmetric power*** $\operatorname{Sym}^m M$ *is the quotient of the tensor power $M^{\otimes m}$ by the submodule generated by elements of the form*

$$m_1\otimes\cdots\otimes m_i\otimes m_{i+1}\otimes\cdots\otimes m_m - m_1\otimes\cdots\otimes m_{i+1}\otimes m_i\otimes\cdots\otimes m_m.$$

So in $\operatorname{Sym}^m M$ the tensor factors commute. We write $m_1\cdots m_m$ instead of $m_1\otimes\cdots\otimes m_m$. There is also a universal property for symmetric powers. Many properties of the symmetric powers can be proved using this.

Theorem 3.2.2 (Universal Property)**.** *Let M,N be R-modules. Then symmetric multilinear maps $M^m\to N$ are the same as R-module homomorphisms $\operatorname{Sym}^m M\to N$.*

Proposition 3.2.3. *Let V be a vector space over k with basis $\{e_1,\dots,e_n\}$, then $\{e_{i_1}\cdots e_{i_m} \mid 1\le i_j\le i_{j+1}\le n\}$ is a basis of $\operatorname{Sym}^m V$. As a result, $\dim_k \operatorname{Sym}^m V = \binom{n+m-1}{m}$. We also have*

$$\operatorname{Sym}^m V \xrightarrow{\sim} k[x_1,\dots,x_n]_m, \quad e_i\mapsto x_i$$

where $k[x_1,\dots,x_n]_m$ is the degree m part of the polynomial ring.

There is another way to construct symmetric tensors. For any $m_1 \otimes \cdots \otimes m_m$, simply sum up all of its permutations. Then we get $\sum_{\sigma \in S_m} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(m)}$, which is clearly invariant under permutations. As one can expect,

Proposition 3.2.4. *If $M \cong R^l$ is a free module, then we have an injection*

$$\operatorname{Sym}^n M \hookrightarrow M^{\otimes n}, \quad m_1 \cdots m_n \mapsto \sum_{\sigma \in S_n} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}.$$

In particular this is true for vector spaces. If $n!$ is invertible in R, then the following map is also injective:

$$\operatorname{Sym}^n M \hookrightarrow M^{\otimes n}, \quad m_1 \cdots m_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}.$$

Moreover, the composition of this map with the projection $M^{\otimes n} \twoheadrightarrow \operatorname{Sym}^n M$ is the identity on $\operatorname{Sym}^m M$.

We can also form the **symmetric algebra** $\operatorname{Sym} M := R \oplus M \oplus \operatorname{Sym}^2 M \oplus \cdots$. This is the "largest" commutative algebra containing M. An insightful fact is that $\operatorname{Sym}(V) \cong k[V^*]$ canonically for a vector space V.

Recall that if $A \in \operatorname{End}_k(V)$, then $A^{\otimes m}$ acts on $V^{\otimes m}$ by $v_1 \otimes \cdots \otimes v_m \mapsto Av_1 \otimes \cdots \otimes Av_m$. So this action passes down to the quotient on $\operatorname{Sym}^m V$. For the same reason, if V is a representation of G, then so is $\operatorname{Sym}^m V$.

Example 3.2.5. $\mathbf{GL}_2(\mathbb{C}) \circlearrowright \mathbb{C}^2$ *naturally. So* $\mathbf{GL}_2(\mathbb{C}) \circlearrowright \operatorname{Sym}^2(\mathbb{C}^2) \cong k[x, y]_2$. *Explicitly,*

$$\rho(g)(ax^2 + bxy + cy^2) = a(gx)^2 + b(gx)(gy) + c(gy)^2.$$

Under the basis $\{x^2, xy, y^2\}$, we can write down the matrix for $\rho(g)$:

$$\rho \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} g_{11}^2 & g_{11}g_{12} & g_{12}^2 \\ 2g_{11}g_{21} & g_{11}g_{22} + g_{12}g_{21} & 2g_{12}g_{21} \\ g_{21}^2 & g_{21}g_{22} & g_{22}^2 \end{bmatrix}$$

which is a polynomial representation.

Lastly, we have the following decomposition.

Proposition 3.2.6. *If V, W are finite dimensional vector spaces, then we have a canonical isomorphism*

$$\operatorname{Sym}^n(V \oplus W) \xrightarrow{\sim} \bigoplus_{a=0}^{n} \operatorname{Sym}^a V \otimes \operatorname{Sym}^{n-a} W,$$

$$v_1 \cdots v_a w_1 \cdots w_{n-a} \mapsfrom (v_1 \cdots v_a) \otimes (w_1 \cdots w_{n-a}).$$

Proof. The map is defined in a coordinate-free way, so it is canonical. To see that it is an isomorphism, note that if $\{e_i\}, \{f_j\}$ are basis of V, W respectively, then we have a correspondence of basis

$$e_{i_1} \cdots e_{i_a} f_{j_1} \cdots f_{j_{n-a}} \leftarrow\!\shortmid (e_{i_1} \cdots e_{i_a}) \otimes (f_{j_1} \cdots f_{j_{n-a}})$$

for all $0 \leq a \leq n, 1 \leq i_1 \leq \cdots \leq i_a \leq a, 1 \leq j_1 \leq \cdots \leq n-a$. ☐

Corollary 3.2.7. *If V, W are also representations of G, then the above decomposition of vector spaces is also a decomposition of representations.*

Proof. We only need that the isomorphism is G-equivariant. Indeed, the following diagram commute:

$$\begin{array}{ccc}
\operatorname{Sym}^n(V \oplus W) & \longleftarrow & \bigoplus_{a=0}^n \operatorname{Sym}^a V \otimes \operatorname{Sym}^{n-a} W \\
\big\downarrow g & & \big\downarrow g \\
\operatorname{Sym}^n(V \oplus W) & \longleftarrow & \bigoplus_{a=0}^n \operatorname{Sym}^a V \otimes \operatorname{Sym}^{n-a} W
\end{array}$$

$$\begin{array}{ccc}
e_{i_1} \cdots e_{i_a} f_{j_1} \cdots f_{j_{n-a}} & \longleftarrow\!\shortmid & (e_{i_1} \cdots e_{i_a}) \otimes (f_{j_1} \cdots f_{j_{n-a}}) \\
\big\downarrow g & & \big\downarrow g \\
(ge_{i_1}) \cdots (ge_{i_a})(gf_{j_1}) \cdots (gf_{j_{n-a}}) & \longleftarrow\!\shortmid & (ge_{i_1} \cdots ge_{i_a}) \otimes (gf_{j_1} \cdots gf_{j_{n-a}})
\end{array}$$

☐

3.3 Exterior products

In the exterior power we require instead that tensors anti-commute, i.e., we want $m \otimes m' = -m' \otimes m$.

Definition 3.3.1. *Let R be commutative, M an R-module. The* ***exterior power*** *$\bigwedge^m M$ is the quotient of the tensor power $M^{\otimes m}$ by the ideal generated by elements of the form*

$$m_1 \otimes \cdots \otimes m_i \otimes m_{i+1} \otimes \cdots \otimes m_m + m_1 \otimes \cdots \otimes m_{i+1} \otimes m_i \otimes \cdots \otimes m_m.$$

We shall write $m_1 \wedge \cdots \wedge m_m$ instead of $m_1 \otimes \cdots \otimes m_m$.

There are parallel results for exterior products.

Theorem 3.3.2 (Universal Property)**.** *Let M, N be R-modules. Then alternating multilinear maps $M^m \to N$ are the same as R-module homomorphisms $\bigwedge^m M \to N$.*

Proposition 3.3.3. *Let V be a vector space over k with basis $\{e_1, \ldots, e_n\}$, then $\{e_{i_1} \cdots e_{i_m} \mid 1 < i_j < i_{j+1} \leq n\}$ is a basis of $\bigwedge^m V$. As a result, $\dim_k \bigwedge^m V = \binom{n}{m}$ for $m \leq n$ and 0 for $m > n$.*

Proposition 3.3.4. *If $M \cong R^l$ is a free module, then we have an injection*

$$\bigwedge^n M \hookrightarrow M^{\otimes n}, \quad m_1 \wedge \cdots \wedge m_n \mapsto \sum_{\sigma \in S_n} \varepsilon(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}.$$

In particular this is true for vector spaces. If $n!$ is invertible in R, then the following map is also injective:

$$\bigwedge^n M \hookrightarrow M^{\otimes n}, \quad m_1 \wedge \cdots \wedge m_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}.$$

Moreover, the composition of the latter map with the projection $M^{\otimes n} \twoheadrightarrow \bigwedge^n M$ is the identity on $\bigwedge^m M$.

Proposition 3.3.5. *If V, W are finite dimensional vector spaces, then we have a canonical isomorphism*

$$\bigwedge^n (V \oplus W) \xrightarrow{\sim} \bigoplus_{a=0}^{n} \bigwedge^a V \otimes \bigwedge^{n-a} W,$$

$$v_1 \wedge \cdots \wedge v_a \wedge w_1 \wedge \cdots \wedge w_{n-a} \leftarrow\!\shortmid (v_1 \wedge \cdots \wedge v_a) \otimes (w_1 \wedge \cdots \wedge w_{n-a}).$$

If V, W are representations of G, then the above is a decomposition of representations.

3.4 Determinant

It is worth mentioning that the exterior powers can be used to develop a coordinate-free version of linear algebra. The determinant is one of the instances.

Let V be an n-dimensional vector space over k. Notice that $\dim_k \bigwedge^n V = \binom{n}{n} = 1$, so for any linear map $A \in \operatorname{End}_k V$, its exterior product $\bigwedge^n A \colon \bigwedge^n V \to \bigwedge^n V$ is canonically identified with a number. If we choose a basis and write down the formula for $\bigwedge^n A$, we will see that this number is exactly $\det A$.

4 Character theory

Talk by Sam Swain, notes by Peter McDonald

Recall, a representation of a finite group G is a homomorphism $\rho : G \to \mathbf{GL}(V)$ where V is a vector space. We often say that V is a representation.

Definition 4.0.1. *A* ***subrepresentation*** *of a group G is a subspace W of a representation V such that $\rho(g)(W) \subset W$ for all $g \in G$.*

Definition 4.0.2. *A representation is called* ***irreducible*** *if its only subrepresentations are 0 and itself.*

4.1 Decomposition of Representations

Let $W \subset V$ be a subrepresentation of a representation V of a group G. Since G is finite, then W has a complement W' such that $V = W \oplus W'$. Let $\pi : V \to W$ be the projection map. Then we can define

$$\pi' := \frac{1}{|G|} \sum_{g \in G} \rho(g) \pi \rho(g^{-1})$$

If we take $x \in W$ then $\rho(g^{-1})(x) \in W$ because W is a subrepresentation. So

$$\pi \rho(g^{-1}) x = \rho(g^{-1}) x$$

because π is a projection. Then

$$\rho(g) \pi \rho(g^{-1}) x = \rho(g) \rho(g^{-1}) x = x$$

Then

$$\pi'(x) = \frac{1}{|G|} \sum_{g \in G} x = x$$

so π' is a projection.

We now show that $\rho(g)\pi' = \pi' \rho(g)$ for all $g \in G$. Note that

$$\begin{aligned} \rho(h)\pi'\rho(h^{-1}) &= \frac{1}{|G|} \sum_{g \in G} \rho(h)\rho(g)\pi'\rho(g^{-1})\rho(h^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(hg)\pi'\rho((hg)^{-1}) \\ &= \pi' \end{aligned}$$

Take $W^{\perp} = \ker \pi'$. Take $x \in W^{\perp}$. Then $\pi'\rho(g)(x) = \rho(g)\pi'(x) = 0 \in W^{\perp}$. Then $W^{\perp}$ is a subrepresentation. So $V = W \oplus W^{\perp}$. We should note that this decomposition does not depend on our choice of π.

If we keep breaking up V into these direct summands, we can eventually write $V = V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}$. Note, this is not always the case when our group is infinite. Consider the following example:

Example 4.1.1. *Consider $\rho : \mathbb{R} \to GL(\mathbb{C}^2)$ given by $\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Consider the subspace spanned by*

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This is invariant under ρ and so is a subrepresentation. However, its complement would be the subspace spanned by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

but $\mathbb{R}$ acts on this subgroup nontrivially.

Proposition 4.1.2 (Schur's Lemma)**.** *Let V and W be irreducible representations. Let $\varphi : V \to W$ be an intertwining operator, i.e., the following diagram commutes*

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \rho_1(g)\Big\downarrow & & \Big\downarrow\rho_2(g) \\ V & \xrightarrow{\varphi} & W \end{array}$$

Then

1. *φ is either the 0 map or an isomorphism*
2. *If φ is an isomorphism, then $\varphi = \lambda Id$.*

Proof. 1. We claim that $\ker\varphi$ and $\mathrm{Im}\varphi$ are subrepresentations of V and W respectively. Take $x \in \ker\varphi$. Then $\rho_2(g)\varphi(x) = 0 = \varphi\rho_1(g)(x)$ which means that $\rho_1(g) \in \ker\varphi$ and therefore $\ker\varphi$ is a subrepresentation. Similarly, $w \in \mathrm{Im}\varphi$ implies there is some $v \in V$ such that $\varphi(v) = w$. Then $\rho_2(g)(w) = \varphi(\rho_1(g)(v))$ which means that $\rho_2(g)(w) \in \mathrm{Im}\varphi$ and $\mathrm{Im}\varphi$ is a subrepresentation. By irreducibility, the only possibilities are $\ker\varphi = 0, \mathrm{Im}\varphi = W$ or $\ker\varphi = V, \mathrm{Im}\varphi = 0$.

2. Assume $V = k^n$ where k is an algebraically closed field. Then φ has an eigenvalue λ in k. Then $\det(\varphi - \lambda\mathrm{Id}) = 0$ means $\ker(\varphi - \lambda\mathrm{Id})$ is nontrivial. Then by (1) $\ker(\varphi - \lambda\mathrm{Id}) = V$ and $\varphi = \lambda\mathrm{Id}$.

□

If we have an isomorphism between representations $\varphi : V \to W$ with

$$V = V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}$$
$$W = W_1^{\oplus n_1} \oplus \cdots \oplus W_j^{\oplus n_j}$$

then we can restrict φ to each irreducible component.

4.2 Character Theory

Another way of extracting irreducible representations from other representations is through the use of characters.

Definition 4.2.1. *If G is a finite group and $\rho : G \to \mathbf{GL}(V)$ is a representation, the **character** of V, denoted χ_V, is a map $\chi_V : G \to k$ given by $\chi_V(g) = tr(\rho(g))$.*

4.2.1 Properties of χ

1. $\chi_V(e) = \dim(V)$.
2. $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.
3. $\chi_V(xgx^{-1}) = \chi_V(g)$

Proof. 1. Since ρ is a representation, we have $\rho(e) = \text{Id}$. Hence, $\chi(e) = \text{tr}(\text{Id}) = \dim(V)$

2.
$$\overline{\chi_V(g)} = \overline{\text{tr}(\rho(g))} = \sum \overline{\lambda_i}$$

We claim this is equal to $\sum \lambda_i^{-1}$. To see this, $\rho(g)^{|G|} = \rho(g^{|G|}) = \rho(e)$ has eigenvalues 1. We also know that the eigenvalues of A^k are the eigenvalues of A raised to the k. Then $|\lambda_i| = 1$ for all $\rho(g)$ and so $\overline{\lambda_i} = \lambda_i^{-1}$. Hence

$$\overline{\chi_V(g)} = \overline{\text{tr}(\rho(g))} = \sum \overline{\lambda_i} = \sum \lambda_i^{-1} = \text{tr}(\rho(g^{-1})) = \chi_V(g^{-1})$$

3. $\text{tr}(AB) = \text{tr}(BA)$ so $\text{tr}(ABA^{-1}) = \text{tr}(B)$.

□

4.2.2 More Properties of χ

If V and W are two representations with respective characters χ_V and χ_W, then

1. $\chi_{V \oplus W} = \chi_V + \chi_W$
2. $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

Proof. 1. $\chi_{V\oplus W}(g)$ is just the trace of the following matrix:

$$\begin{pmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{pmatrix}$$

Then the trace of this matrix is just $\text{tr}(\rho_V) + \text{tr}(\rho_W) = \chi_V(g) + \chi_W(g)$.

2. Look at the Kronecker product

□

5 Representations of the symmetric group

Talk by Faith Pearson, notes by Faith Pearson

(Additional proofs and examples are covered in these notes that were not covered in the talk.)

Our goal in this section is to classify all of the irreducible representations of the symmetric group. Given any finite group G, the number of irreducible representations of G is equal to the number of conjugacy classes of G. Though this is true for every finite group G, it is not always possible to create an explicit bijection between its irreducible representations and conjugacy classes. However, we will see that each irreducible representation of the symmetric group has a one-to-one correspondence with a combinatorial object called a Young diagram that corresponds to a given conjugacy class.

In order to study the irreducible representations of the symmetric group, we must first establish its conjugacy classes.

Definition 5.0.1. *Suppose $\sigma \in S_n$ is a product of cycles $\sigma_1\sigma_2\cdots\sigma_k$ and let λ_i be the length of σ_i. We may assume that $\lambda_i \geq \lambda_{i+1}$ for all i since disjoint cycles commute. Then we define $\lambda(\sigma) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ to be the* ***cycle type*** *of σ.*

Definition 5.0.2. *For a given $\sigma \in S_n$, the* ***conjugacy class*** *C_σ is the set of all elements $\sigma' = \tau\sigma\tau^{-1}$ for some $\tau \in S_n$.*

We can now determine the conjugacy classes of the symmetric group S_n. We start by noticing that any conjugate of a k-cycle is also a k-cycle.

Lemma 5.0.3. *Let $\alpha, \tau \in S_n$, where α is a k-cycle $(a_1, a_2, \ldots, a_k)$. Then*

$$\tau\alpha\tau^{-1} = (\tau(a_1), \tau(a_2), \ldots, \tau(a_k)).$$

Proof. Consider $\tau(a_i)$ such that $1 \leq i \leq k$. Then we have $\tau^{-1}\tau(a_i) = a_i$, and $\alpha(a_i) = a_{i+1} \mod k$. Then $\tau\alpha\tau^{-1}(\tau(a_i)) = \tau(a_{i+1}) \mod k$. Now take any j where $j \in \{1, 2, \ldots, n\}$, and where $j \neq a_i$ for any i. Then $\alpha(j) = j$ because j is not in the k-cycle defining α, and so $\tau\alpha\tau^{-1}(\tau(j)) = \tau(j)$. Hence, $\tau\alpha\tau^{-1}$ fixes any number which is not of the form $\tau(a_i)$ for some i, and thus

$$\tau\alpha\tau^{-1} = (\tau(a_1), \tau(a_2), \ldots \tau(a_k)).$$

□

Lemma 5.0.4. *The conjugate of a product of k-cycles is equivalent to the product of the conjugates of k-cycles. That is, for α_i disjoint, we have that*

$$\tau\alpha_1\alpha_2\ldots\alpha_n\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_2\tau^{-1})\ldots(\tau\alpha_n\tau^{-1}).$$

Finally, we can describe the conjugacy classes of the symmetric group.

Proposition 5.0.5. *The conjugacy classes of S_n are determined by cycle type. That is, if σ has cycle type $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, and if ρ is any other element of S_n with cycle type $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, then σ is conjugate to ρ.*

Proof. Suppose that σ is a product of disjoint cycles $\sigma = \alpha_1\alpha_2\ldots\alpha_\ell$, with cycle type $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where α_i is a λ_i-cycle. By Lemma 5.0.4,

$$\tau\sigma\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_2\tau^{-1})\ldots(\tau\alpha_n\tau^{-1}),$$

and by Lemma 5.0.3, $\tau\alpha_i\tau^{-1}$ is a λ_i-cycle. For any $i, j \in \{1, 2, \ldots, n\}$ where $i \neq j$, α_i and α_j are disjoint. Then since τ is a bijection, this implies $\tau\alpha_i\tau^{-1}$ and $\tau\alpha_j\tau^{-1}$ must also be disjoint. Thus the conjugate $\tau\sigma\tau^{-1}$ is a product of disjoint λ_i-cycles, and has cycle type $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.

Conversely, suppose σ and ρ both have cycle type $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. Let $\sigma = \alpha_1\alpha_2\ldots\alpha_\ell$ and let $\rho = \beta_1\beta_2\ldots\beta_\ell$, where α_i and β_i are λ_i cycles. Then $\alpha_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,\lambda_i})$ and $b_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,\lambda_i})$. We can chose τ to be the map such that $\tau(a_{i,k}) = b_{i,k}$. Because α_i are mutually disjoint and similarly β_i are mutually disjoint, and σ and ρ are permutations on $\{1, 2, \ldots, n\}$, τ is well defined and also a permutation. Thus by Lemma 5.0.3, $\tau\sigma\tau^{-1} = \rho$.

□

Thus, we see that any two permutations in S_n are conjugate if and only if they have the same cycle type.

Recall the definition of the group algebra, so we can define a few new representations.

Definition 5.0.6. *The **group algebra** $\mathbb{C}S_n$ is the set of all finite formal sums of the form*

$$\sum_{\sigma \in S_n} z_\sigma e_\sigma \text{ for } z_\sigma \in \mathbb{C}$$

where e_σ are basis elements indexed by $\sigma \in S_n$. Multiplication is defined on basis elements by

$$\left(\sum z_i e_{\sigma_i}\right)\left(\sum y_j e_{\sigma_j}\right) = \sum z_i y_j e_{\sigma_i} e_{\sigma_j}$$

which we can expand linearly.

Definition 5.0.7. *We may also think of $\mathbb{C}S_n$ as a complex vector space with basis elements indexed by each e_σ. Then we can define a representation*

$$\rho : S_n \to GL(\mathbb{C}S_n) \cong GL(\mathbb{C}^{n!}),$$

*which is called the **regular representation**.*

Let us define two more representations that will come up again in a later example.

Definition 5.0.8. *For S_n, the **alternating representation** (or sign representation) is $\mathbb{C}$ equipped with the action*

$$\sigma \cdot v = \begin{cases} v, & \text{if } \sigma \text{ is an even permutation} \\ -v, & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

or equivalently, $\rho(\sigma) = sgn(\sigma)I$ for every $\sigma \in S_n$. Remark that any S_n where $n \geq 2$ has the alternating representation, and since this representation is one dimensional, it is irreducible.

Definition 5.0.9. *For any n, let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$, and define the action of S_n on $\mathbb{C}^n$ to be*

$$\sigma(a_1e_1 + a_2e_2 + \cdot + a_ne_n) = a_1e_{\sigma(1)} + a_2e_{\sigma(2)} + \cdot + a_ne_{\sigma(n)}.$$

*This is a permutation representation of S_n. Remark that the one-dimensional subspace of $\mathbb{C}$ spanned by $e_1 + e_2 + \cdots + e_n$ is invariant under the action of S_n, and so its orthogonal complement $V = \{(x_1, x_2, \ldots, x_n) | x_1 + x_2 + \cdots + x_n = 0\}$ is also invariant, and therefore a subrepresentation. We call V the **standard representation** of S_n.*

Definition 5.0.10. *A **partition** of a positive integer $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ is an ordered set $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\lambda_i \geq \lambda_{i+1}$ for every $1 \leq i \leq k$.*

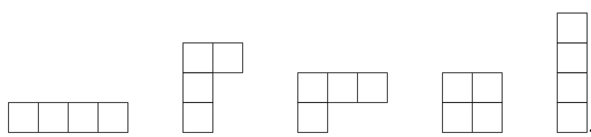
Recall that in S_n, the number of its irreducible representations is equal to the number of its conjugacy classes, and each conjugacy class is a cycle-type equivalence class. Now, there is also a bijective correspondence between the set of cycle-types and the ways n can be written as the sum of positive integers. For example, S_4 has the following five cycle-types:

Cycle Notation	Alternate Form	Corresponding Sum
ϵ	(1)(2)(3)(4)	1+1+1+1
(1 2)	(1 2)(3)(4)	2+1+1
(1 2 3)	(1 2 3)(4)	3+1
(1 2)(3 4)	(1 2)(3 4)	2+2
(1 2 3 4)	(1 2 3 4)	4

A fundamental tool for studying the representations of S_n is the Young diagram.

Definition 5.0.11. *A **Young diagram** is a graphical representation of a partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$*, as an array of boxes. This array is constructed by drawing a row of* λ_1 *boxes, then beneath it drawing a row of* λ_2 *boxes, and so on until the last row contains* λ_k *boxes, and each row is as long as or shorter than the one above it.*

Example 5.0.12. *The Young Diagrams corresponding to the partitions of* $n = 4$ *are:*

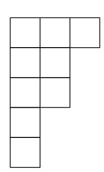


So we see each conjugacy class of S_n corresponds to a Young diagram, and now we can find a method that will generate all of the irreducible representations of S_n. (For a proof of why this method works, see Section 4.2 of Fulton and Harris.) To continue, we must define a Young tableau.

Definition 5.0.13. *A **Young tableau** is a Young diagram whose boxes are labeled in any way with each of the numbers* $1, \ldots, n$.

For our purposes, we will fill in our Young diagrams in the natural way, starting with 1 in the upper left and increasing by 1 as we move from left to right and top to bottom.

Example 5.0.14. *For* $n = 9$*, the partition* $\lambda = (3, 2, 2, 1, 1)$ *induces the Young diagram,*



and its associated Young tableau would be

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1} 9 \\ \cline{1-1}\end{array}\ .$$

Given any Young tableau, we can define

$$P_\lambda = \{\sigma \in S_n \mid \sigma \text{ preserves each row}\}, \text{ and}$$
$$Q_\lambda = \{\sigma \in S_n \mid \sigma \text{ preserves each column}\}.$$

Example 5.0.15. *If we were working in S_6, and wanted to find the irreducible representations corresponding to the partition $\lambda = (3, 2, 1)$, our Young tableau would be*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\ .$$

Then we would have

$$P_\lambda = \{e, (12), (23), (13), (123), (132), (45), (12)(45), (23)(45), (13)(45),$$
$$(123)(45), (132)(45)\},$$
$$Q_\lambda = \{e, (14), (16), (46), (146), (164), (25), (14)(25), (16)(25), (46)(25),$$
$$(146)(25), (164)(25)\}.$$

We can use P_λ and Q_λ to define the elements a_λ, b_λ in the group algebra $\mathbb{C}S_n$ to be

$$a_\lambda := \sum_{\sigma \in P_\lambda} e_\sigma$$
$$b_\lambda := \sum_{\sigma \in Q_\lambda} \operatorname{sgn}(\sigma) e_\sigma$$

Finally, we can define a key tool in finding the irreducible representations of S_n.

Definition 5.0.16. *The* ***Young symmetrizer*** *is*

$$c_\lambda := a_\lambda \cdot b_\lambda.$$

Theorem 5.0.17. *Given S_n, let λ be a partition of n. Let $V_\lambda = \mathbb{C}S_n \cdot c_\lambda$ be the subspace of $\mathbb{C}S_n$ spanned by the Young symmetrizer c_λ. Then*

1. *V_λ is an irreducible representation of S_n.*

2. *If λ and μ are distinct partitions of n, then V_λ and V_μ are not isomorphic.*

3. *The V_λ account for all of the irreducible representations of S_n.*

We will follow Sean McAfee's proof. The full proof can be found in Sections 4.1 and 4.2 in Fulton and Harris.

Lemma 5.0.18. *For all $x \in \mathbb{C}S_n$, $c_\lambda \cdot x \cdot c_\lambda$ is a scalar multiple of c_λ.*

Lemma 5.0.19. *If $\lambda \neq \mu$, then $c_\lambda \cdot \mathbb{C}S_n \cdot c_\mu = 0$.*

Assuming these two lemmas to be true, we can now prove Theorem 2.8.

Proof. 1. Take $V_\lambda = \mathbb{C}S_n \cdot c_\lambda$ for a given Young symmetrizer c_λ. Then by Lemma 2.9, we have that

$$c_\lambda V_\lambda \subseteq \mathbb{C}c_\lambda.$$

Let W be a nonzero subrepresentation of V_λ. We want to show $W = V_\lambda$. First, we claim that $c_\lambda V_\lambda$ and $c_\lambda W$ are both nonzero. Suppose that $c_\lambda V_\lambda = 0$. Then

$$V_\lambda V_\lambda = \mathbb{C}S_n(c_\lambda V_\lambda) = 0.$$

Considering $\mathbb{C}S_n$ and $\mathbb{C}S_n \cdot c_\lambda$ as subspaces, there exists a projection $\pi : \mathbb{C}S_n \to \mathbb{C}S_n \cdot c_\lambda$ that commutes with the action of S_n. This projection can be described as right multiplication on the group algebra $\mathbb{C}S_n$ by an element $x \in \mathbb{C}S_n$ by letting $x := \pi(1)$. Since $x = 1 \cdot x$, this x must be in V_λ. Then by the definition of projection,

$$x = x^2 \in V_\lambda V_\lambda = 0,$$

and thus x must equal zero, which is a contradiction since the nonzero c_λ itself is in $\mathbb{C}S_n \cdot c_\lambda$. Therefore we must have $c_\lambda V_\lambda \neq 0$. Next we will show $c_\lambda W \neq 0$.

Since we have that W is a subspace of V_λ, $c_\lambda V_\lambda \subseteq \mathbb{C}c_\lambda$, and that $c_\lambda W \neq 0$, we must have $c_\lambda W = \mathbb{C}c_\lambda$. Therefore,

$$V_\lambda = \mathbb{C}S_n \cdot c_\lambda = \mathbb{C}S_n(\mathbb{C}Cc_\lambda) = \mathbb{C}S_n(c_\lambda W) \subseteq W$$

where the inclusion on the right follows from the fact that W is a subrepresentation of V_λ, that is, W is invariant under the action of $\mathbb{C}S_n$. Thus, we have $V_\lambda = W$, which completes the proof that V_λ is irreducible in S_n.

2. Let λ and μ be distinct partitions of n, and let V_λ and V_μ be their corresponding representations. By (a), we have that $c_\lambda V_\lambda = \mathbb{C}c_\lambda \neq 0$, and by Lemma 2.10, we have that

$$c_\lambda V_\mu = c_\lambda \mathbb{C}S_n c_\mu = 0.$$

Thus, V_λ and V_μ cannot be isomorphic, and therefore, if λ and μ are distinct partitions of n, then V_λ and V_μ are not isomorphic.

3. We know that each partition λ of n is in one-to-one correspondence with a distinct conjugacy class of S_n. We know from (b) that the V_λ determine by such partitions are all inequivalent. Then since the number of conjugacy classes of a finite group is equal to the number of irreducible representations, we have therefore accounted for all of the irreducible representations of S_n.

$\square$

Therefore we have proved that the subspace $\mathbb{C}S_n \cdot c_\lambda$ is an irreducible representation of S_n, and distinct partitions of λ correspond to distinct irreducible representations. Furthermore, the $\mathbb{C}S_n \cdot c_\lambda$ account for all irreducible representations of S_n.

Example 5.0.20. *Let us use what we have learned to find all of the irreducible representations of S_3. There are three Young diagrams corresponding to the three partitions $\lambda = (3)$, $\mu = (2,1)$, $\omega = (1,1,1)$, which are respectively*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}.$$

In the first Young tableau, since all of the numbers are in the same row, any permutation will preserve the row. However, the only permutation that will preserve the columns is the identity. Thus, $P_\lambda = S_3$, and $Q_\lambda = \epsilon$. So we have

$$\begin{aligned} a_\lambda &= e_\epsilon + e_{(12)} + e_{(23)} + e_{(13)} + e_{(123)} + e_{(132)}, \\ b_\lambda &= e_\epsilon, \\ c_\lambda &= (e_\epsilon + e_{(12)} + e_{(23)} + e_{(13)} + e_{(123)} + e_{(132)})e_\epsilon \\ &= e_\epsilon + e_{(12)} + e_{(23)} + e_{(13)} + e_{(123)} + e_{(132)}. \end{aligned} \tag{1}$$

Therefore $\mathbb{C}S_3 \cdot c_\lambda = \mathbb{C} \cdot c_\lambda = \langle c_\lambda \rangle$ is the associated irreducible representation since multiplying by any element in the basis of $\mathbb{C}S_3$ will simply rearrange the addends of c_λ, however it will not change the sum. Remark that the subspace generated by c_λ is one dimensional, and because $\sigma \cdot rc_\lambda = rc_\lambda$ for any $\sigma \in S_3$ and any $r \in \mathbb{C}$, the action of every σ leaves every vector in $\langle c_\lambda \rangle$ fixed. Therefore $\langle c_\lambda \rangle$ is the trivial representation.

In the second Young diagram, we have $P_\mu = \{\epsilon, (12)\}$, and $Q_\mu = \{\epsilon, (13)\}$. Then we obtain

$$\begin{aligned} a_\mu &= e_\epsilon + e_{(12)}, \\ b_\mu &= e_\epsilon - e_{(13)}, \\ c_\mu &= (e_\epsilon + e_{(12)})(e_\epsilon - e_{(13)}) \\ &= e_\epsilon - e_{(13)} + e_{(12)} - e_{(132)}. \end{aligned} \tag{2}$$

The associated irreducible representation is $\mathbb{C}S_3 \cdot c_\mu$. To find out what this subspace is, we multiply c_μ by the basis elements of $\mathbb{C}S_3$, and obtain

$$
\begin{aligned}
e_\epsilon(e_\epsilon - e_{(13)} + e_{(12)} - e_{(132)}) &= e_\epsilon - e_{(13)} + e_{(12)} - e_{(132)}, \\
e_{(12)}(e_\epsilon - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(12)} - e_{(132)} + e_\epsilon - e_{(13)}, \\
e_{(13)}(e_\epsilon - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(13)} - e_\epsilon + e_{(123)} - e_{(23)} \\
e_{(23)}(e_\epsilon - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(23)} - e_{(123)} + e_{(132)} - e_{(12)} \\
e_{(123)}(e_\epsilon - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(123)} - e_{(23)} + e_{(13)} - e_\epsilon \\
e_{(132)}(e_\epsilon - e_{(13)} + e_{(12)} - e_{(132)}) &= e_{(132)} - e_{(12)} + e_{(23)} - e_{(123)}.
\end{aligned} \tag{3}
$$

The matrix corresponding to the above set of equations is

$$
\begin{pmatrix}
1 & 1 & -1 & 0 & -1 & 0 \\
1 & 1 & 0 & -1 & 0 & -1 \\
-1 & -1 & 1 & 0 & 1 & 0 \\
0 & 0 & -1 & 1 & -1 & 1 \\
0 & 0 & 1 & -1 & 1 & -1 \\
-1 & -1 & 0 & 1 & 0 & 1
\end{pmatrix}.
$$

Since the set is spanned by the first and third vectors, $\mathbb{C}S_3 \cdot c_\mu$ is the subspace

$$
\langle e_\epsilon - e_{(13)} + e_{(12)} - e_{(132)},\, e_{(13)} - e_\epsilon + e_{(123)} - e_{(23)} \rangle.
$$

This must be the standard representation, since it is the only two-dimensional representation of S_3.

In the third Young diagram, any permutation in S_3 will preserve the column, however only the identity will fix the rows. So we have $P_\omega = \{\epsilon\}$, and $Q_\omega = S_3$. Then we obtain

$$
\begin{aligned}
a_\omega &= e_\epsilon, \\
b_\omega &= e_\epsilon - e_{(12)} - e_{(23)} - e_{(13)} + e_{(123)} + e_{(132)}, \\
c_\omega &= e_\epsilon(e_\epsilon - e_{(12)} - e_{(23)} - e_{(13)} + e_{(123)} + e_{(132)}) \\
&= e_\epsilon - e_{(12)} - e_{(23)} - e_{(13)} + e_{(123)} + e_{(132)}.
\end{aligned} \tag{4}
$$

Once again, $\mathbb{C}S_3 \cdot c_\omega = \mathbb{C} \cdot c_\omega = \langle c_\omega \rangle$ is the associated irreducible representation since multiplying by any element in the basis of $\mathbb{C}S_3$ will rearrange the addends of c_ω and negate their signs. This subspace is also one dimensional, and for any $\sigma \in S_3$, and any $r \in \mathbb{C}$, we have $\sigma \cdot rc_\omega = rc_\omega$ if σ is even, and $\sigma \cdot rc_\omega = -rc_\omega$ if σ is odd. Thus, $\langle c_\omega \rangle$ is the alternating representation.

Thus, the three representations of S_3 are the trivial representation, the standard representation, and the alternating representation. This method will still hold for any symmetric group, however the Young symmetrizers size become large very quickly.

6 Double Centralizer Theorem

Talk by Sabine Lang, notes by Peter McDonald

In this section, let k be an algebraically closed field, A a finite dimensional k-algebra, and $\rho : A \to \mathrm{End}(V)$ is a representation of A.

Definition 6.0.1. *The **radical of** A is the set*

$$Rad(A) := \{x \in A : xM = 0 \ \forall M \ \textit{irreducible left } A\textit{-modules}\}$$

*If $Rad(A) = 0$ we say that A is **semisimple**.*

Theorem 6.0.2. *Up to isomorphism A has finitely many irreducible representations V_i with $\dim(V_i) < \infty$ for all i and $A/Rad(A) \cong \bigoplus_{i=1}^n \mathrm{End}(V_i)$*

Proof. Let V_i be an irreducible representation of A. If $0 \neq v \in V_i$, then $0 \neq Av \subseteq V_i$, hence $Av = V_i$. So $\dim(V_i) = \dim(Av) < \infty$ because A is finite dimensional. By linear algebra

$$\bigoplus_{i \in I} \rho_i : A \to \bigoplus_{i \in I} \mathrm{End}(V_i)$$

is surjective. Then

$$|I| = \text{ \# of irreducible representations} \leq \sum_{i \in I} \dim(\mathrm{End}(V_i)) < \dim(A) < \infty$$

Finally,

$$\mathrm{Rad}(A) = \ker\left(\bigoplus_{i \in I} \rho_i\right),$$

so by the first isomorphism theorem

$$A/\mathrm{Rad}(A) \cong \bigoplus_{i=1}^{n} \mathrm{End}(V_i).$$

$\square$

Corollary 6.0.3.

$$\dim(A) - \dim(\mathrm{Rad}(A)) = \sum_{i \in I} \dim^2(V_i) = \sum_{i \in I} \dim \mathrm{End}(V_i).$$

Theorem 6.0.4. *If A is finite dimensional, the following are equivalent.*

1. *A is semisimple*
2. $\sum_{i=1}^n \dim^2(V_i) = \dim(A)$
3. $A \cong \bigoplus_i Mat_{d_i}(k)$

4. *Any finite dimensional representation of A is completely reducible.*

5. *A is completely reducible as a representation of A.*

Definition 6.0.5. *Let E be a finite dimensional k-vector space. Consider* $\mathrm{End}(E)$ *and $A \subseteq \mathrm{End}(E)$ a subalgebra. Then the centralizer of A in* $\mathrm{End}(E)$ *is the set*

$$\begin{aligned}\mathrm{End}_A(E) &:= \{f : E \to E : f \text{ is linear and a morphism of } A\text{-representations}\}\\ &= \text{the centralizer of } A\end{aligned}$$

Theorem 6.0.6 (Double Centralizer Theorem)**.** *Let E be a finite dimensional k-vector space and $A \subseteq \mathrm{End}(E)$ a subalgebra such that A is semisimple. Let $B = \mathrm{End}_A(E)$. Then*

1. $\mathrm{End}_B(E) = A$

2. *B is semisimple*

3. *$E = \bigoplus_{i=1}^n (V_i \otimes W_i)$ as a representation of $A \otimes B$ where the V_i are irreducible representations of A and the W_i are irreducible representations of B. This gives a bijection between the irreducible representations of A and the irreducible representations of B.*

Proof. Let $V_1, \ldots, V_n$ be irreducbile representations of A. Then, because A is semisimple

$$A \cong \bigoplus_{i=1}^n \mathrm{End}(V_i).$$

We can also decompose E using these representations of A:

$$E \cong \bigoplus_{i=1}^n (V_i \otimes \mathrm{Hom}_A(V_i, E))$$

Let

$$W_i = \mathrm{Hom}_A(V_i, E).$$

Then

$$\begin{aligned}
B &= \mathrm{End}_A(E) \\
&= \mathrm{Hom}_A(E, E) \\
&\cong \mathrm{Hom}_A\left(\bigoplus_{i=1}^{n}(V_i \otimes W_i), E\right) \\
&= \bigoplus_{i=1}^{n} \mathrm{Hom}_A(V_i \otimes W_i, E) \\
&= \bigoplus_{i=1}^{n} \mathrm{Hom}_A(W_i \otimes V_i, E) \\
&\cong \bigoplus_{i=1}^{n} \mathrm{Hom}_A(W_i, \mathrm{Hom}_A(V_i, E)) \\
&= \bigoplus_{i=1}^{n} \mathrm{Hom}_A(W_i, W_i) \\
&= \bigoplus_{i=1}^{n} \mathrm{End}_A(W_i)
\end{aligned}$$

We now show that the W_i are irreducible B-modules. Let $f, f' \in W_i = \mathrm{Hom}_A(V_i, E)$. We know V_i is irreducible as an A-module, so given $0 \neq v \in V_i$, $V_i = Av$, which means that f and f' are determined by $f(v)$ and $f'(v)$ respectively. Then $Af(v) \subseteq E$ is an A-invariant subspace and there is an invariant complement W. Then $E = Af(v) \oplus W$.

Let $T : E \to E$ be defined by

$$\begin{aligned}
af(v) &\longmapsto af'(v) \\
W &\longmapsto W
\end{aligned}$$

Then $T \circ f = f'$. T is an A-homomorphism, so $T \in \mathrm{End}_A(E) = B$. Then W_i is an irreducible B-module. Because $B = \bigoplus_{i=1}^{n} \mathrm{End}_A(W_i)$, B is semisimple.

We are now ready to show $\mathrm{End}_B(E) = A$. As B-modules

$$E \cong \bigoplus_{i=1}^{n}(W_i \otimes \mathrm{Hom}_B(W_i, E))$$

We know $W_i = \mathrm{Hom}_A(V_i, E)$ are irreducible B-modules. Comparing the two decomposition of E as

$$E \cong \bigoplus_{i=1}^{n}(W_i \otimes \mathrm{Hom}_B(W_i, E)) \cong \bigoplus_{i=1}^{n} (V_i \otimes W_i)\,,$$

we can deduce that $V_i \cong \mathrm{Hom}_B(W_i, E)$ (Note: this can be seen in several different ways. One of them is to use the decomposition as an A-module, with the fact that each irreducible for A is one of $V_1, \dots, V_n$.). Hence,

$$E \cong \bigoplus_{i=1}^{n} (V_i \otimes W_i)$$

is a decomposition of E as an $A \otimes B$-module. Then

$$\begin{aligned}
\mathrm{End}_B(E) &= \mathrm{Hom}_B(E, E) \\
&= \mathrm{Hom}_B\left(\bigoplus_{i=1}^{n} (V_{\otimes} W_i), E\right) \\
&\cong \bigoplus \mathrm{Hom}(V_i, \mathrm{Hom}_B(W_i, E)) \\
&= \bigoplus_{i=1}^{n} \mathrm{End}(V_i)
\end{aligned}$$

□

7 Schur-Weyl duality

Talk by Chengyu Du, notes by Sam Swain

Recall the Double Centralizer Theorem from last week:

Theorem 7.0.1. *Given a finite-dimensional vector space E and $A \subseteq \mathrm{End}(E)$ semisimple, then defining $B := \mathrm{End}_A(E)$, we have that*

1. $A = \mathrm{End}_B(E)$
2. *B is semisimple*
3. *$E \cong \bigoplus_{i=1}^{r} (V_i \otimes_{\mathbb{C}} W_i)$ where the V_i are irreducible A-modules and the W_i are irreducible B-modules and $W_i = \mathrm{Hom}(V_i, E)$.*

We now consider the special case where $E = V^{\otimes d}$ with $\dim(V) = n$ and

$$A = \mathbb{C}S_d = \bigoplus_{|\lambda|=d} \mathbb{C}S_d c_\lambda$$

Theorem 7.0.2 (Schur-Weyl Duality)**.** *Given $A = \mathbb{C}S_d = \bigoplus_{|\lambda|=d} \mathbb{C}S_d c_\lambda$ and $E = V^{\otimes d}$. Then*

$$E = V^{\otimes d} \cong \bigoplus_{|\lambda|=d} (\mathbb{S}_\lambda V)$$

Proof. A is semi-simple as it is the direct sum of irreducible A-modules. We now define an A-action on $V^{\otimes d}$. Consider $\sigma \in S_d$. Then define

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_d) := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}$$

which we can expand linearly to $V^{\otimes d}$. Then $\dim(D) = d \cdot n < \infty$ which means we can apply the double centralizer theorem.

In this case $B = \mathrm{End}_A(V^{\otimes d})$. Then

$$\begin{aligned} \mathrm{End}_{\mathbb{C}}(V^{\otimes d}) &\cong (V^{\otimes d})^* \otimes_{\mathbb{C}} V^{\otimes d} \\ &\cong (V^*)^{\otimes d} \otimes_{\mathbb{C}} V^{\otimes d} \\ &\cong (V^* \otimes_{\mathbb{C}} V)^{\otimes d} \\ &\cong (\mathrm{End}(V))^{\otimes d} \end{aligned}$$

Then B consists of the elements in $\mathrm{End}(V)^{\otimes d}$ that commute with the S_d action. In general, these elements look like

$$\phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_d, \qquad \phi_i \in \mathrm{End}(V)$$

In fact, it turns out that B is spanned by elements like $\phi \otimes \phi \otimes \cdots \otimes \phi$ (this is not trivial, but was not covered in the talk) so $B \simeq \mathrm{End}(V)$.

Then, letting $V_\lambda = \mathbb{C}S_\lambda \cdot c_\lambda$ we have

$$V^{\otimes d} \cong \bigoplus (V_\lambda \otimes_{\mathbb{C}} \mathrm{Hom}_A(V_\lambda, V^{\otimes d})) \cong \bigoplus_{|\lambda|=d} (V \otimes_{\mathbb{C}} \mathbb{S}_\lambda V)$$

due to the fact that

$$\begin{aligned} \mathrm{Hom}_A(V_\lambda, V^{\otimes d}) &\cong V^{\otimes d} \otimes_A V_\lambda^* \\ &\cong V^{\otimes d} \otimes_A V_\lambda \\ &\cong V^{\otimes d} \otimes_A A \cdot c_\lambda \\ &\cong V^{\otimes d} \cdot c_\lambda \\ &\cong \mathrm{Im} c_\lambda \end{aligned}$$

Note that the transition from the third-to-last to the second-to-last step is not trivial. ☐

We should also note that Fulton-Harris proves this using the fact that we can think of $\bigoplus_{|\lambda|=d} (\mathbb{S}_\lambda V)^{\oplus m_\lambda}$ as an $\mathrm{End}(V)$-module where $m_\lambda = \dim V_\lambda = \dim \mathbb{S}_\lambda V$. However, this loses the A-module structure.

Example 7.0.3. $V = \mathit{span}\{e_1, e_2\}$.

Proof. We will use the irreducible representation of S_3:

$$\begin{aligned} \mathbb{C}S_3 \cdot c_{(3)} &\simeq \mathrm{Sym}^3 V \\ \mathbb{C}S_3 \cdot c_{(2,1)} &\simeq ? \\ \mathbb{C}S_3 \cdot c_{(1,1,1)} &\simeq \Lambda^3 V \end{aligned}$$

Note that $\Lambda^3 V = 0$ because $\dim(V) = 2 < 3$.

Now

$$\mathrm{Im} c_{(3)} = \mathrm{Sym}^3 V = \mathrm{span}\left\{ \begin{array}{c} e_1 \otimes e_1 \otimes e_1, \\ e_2 \otimes e_2 \otimes e_2, \\ e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1, \\ e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \end{array} \right\}$$

so $\dim \mathrm{Sym}^3 V = 4$. Now $\dim V^{\otimes 3} = 8$ and

$$V^{\otimes 3} \cong \bigoplus_{|\lambda|=d} (\mathbb{S}_\lambda V)^{m_\lambda} = \left(\mathrm{Sym}^3 V\right)^1 \oplus ? \oplus 0$$

From Faith's talk, we know $\dim(V_{(2,1)}) = 2$, and after lots of computation we can get

$$\mathrm{Im} c_{(2,1)} = \mathrm{span}\{e_1 \otimes e_1 \otimes e_2 - e_2 \otimes e_1 \otimes e_1, e_2 \otimes e_2 \otimes e_1 - e_1 \otimes e_2 \otimes e_2\}$$

□

8 Determinental rings and applications to commutative algebra

Talk by Jenny Kenkel, notes by Jenny Kenkel

My objects of interest are the ring of a field adjoin nm many variables, and the ideal generated by minors:

$$R = \mathbb{F}\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \dots & x_{mn} \end{bmatrix}, \ I_{r+1} = (\text{ size } r+1 \text{ minors })$$

Note: R/I^{r+1} is setting all size $r+1$ minors to zero, so it corresponds to matrices of rank r or less.

The example I consider the most is:

$$R = \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}, \ I_2 = (\Delta_1 = vz - wy, \Delta_2 = wx - uz, \Delta_3 = uy - vx)$$

Properties of R/I: in general, this ring is *nice but not too nice.*

- I is prime (nice)
- $u\Delta_1 + v\Delta_2 + w\Delta_3 = 0$ and $x\Delta_1 + y\Delta_2 + z\Delta_3 = 0$ (not too nice)
- $\dim(R/I) = 4$ as a *ring*, that is, we can find a chain of prime ideals with four containments (this ring is infinite dimensional as a vector space)

$$(\Delta_1, \Delta_2, \Delta_3) \subsetneq (\Delta_1, \Delta_2, \Delta_3, v - x) \subsetneq (\Delta_1, \Delta_2, \Delta_3, v - x, w - y)$$
$$\subsetneq (\Delta_1, \Delta_2, \Delta_3, v - x, w - y, u - z) \subsetneq (u, v, w, x, y, z)$$

8.1 Determinantal Rings Are Cohen-Macaulay

The following section is a description of why determinantal rings are interesting to algebraists, and an example of the way commutative algebraists often think of rings.

Local Ring Fact: Define the radical of J, denoted $\sqrt{J}$ to be

$$\sqrt{J} = \{r | r^n \in J\}$$

Then I can always find some ideal that is generated by $\dim(S)$ many elements that's radical is the maximal ideal. Those $\dim(S)$ elements are referred to as a system of parameters.

In the case of

$$R = \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}, \ I_2 = (\Delta_1 = vz - wy, \Delta_2 = wx - uz, \Delta_3 = uy - vx)$$

I claim $\sqrt{(u, v-x, w-y, z)} = (u, v, w, x, y, z)$, the maximal (homogenous) ideal of R/I.

Note: the ring R has infinitely many maximal ideals, but only one maximal homogenous ideal. For many purposes, then, it can be considered a local ring.

Proof. Note that $\sqrt{(u, v-x, w-y, z)} \subseteq (u, v, w, x, y, z)$, so it suffices to show that

$$(u, v, w, x, y, z) \subseteq \sqrt{(u, v-x, w-y, z)}$$

For purposes of this proof, let $\mathfrak{n} = (u, v-x, w-y, z)$.

Certainly, $u, z \in \mathfrak{n}$. As $u \in \mathfrak{n}$, we have that $uy \in \mathfrak{n}$. Recall that $uy - vx = 0 \in \mathfrak{n}$, so $vx \in \mathfrak{n}$. Now,

$$v(v-x) = v^2 - vx \in \mathfrak{n} \text{ so } v^2 \in \mathfrak{n} \text{ and similarly}$$
$$x(v-x) = vx - x^2 \in \mathfrak{n} \text{ so } x^2 \in \mathfrak{n}$$

In a symmetric argument, since $z \in \mathfrak{n}, vz \in \mathfrak{n}$ and so $wy \in \mathfrak{n}$. Thus,

$$w(w-y) = w^2 - wy \in \mathfrak{n} \text{ so } w^2 \in \mathfrak{n}$$
$$y(w-y) = wy - y^2 \in \mathfrak{n} \text{ so } y^2 \in \mathfrak{n}$$

Thus, we have shown that u, v^2, w^2, x^2, y^2 and z are in $\mathfrak{n}$, and so u, v, w, x, y and z are in $\sqrt{\mathfrak{n}}$. □

Definition 8.1.1. *Regular Sequence*

A regular sequence in a ring S is a sequence of elements, $x_1, \ldots, x_n$ such that

- x_1 is not a zero divisor in S and $x_1 S \neq S$
- x_2 is not a zero divisor in $S/(x_1)$ and $(x_1, x_2) \neq S$

- x_i is not a zero divisor in $S/(x_1, \dots, x_{i-1}$ and $(x_1, \dots, x_i) \neq S$

A very neat property about the ring R/I is that the system of parameters we discussed above is in fact a regular sequence!
Sketch of Proof that the system of parameters is a regular sequence: Since I is a prime ideal, R/I is a domain, so there are no zero divisors, and in particular, u is not a zero divisor in R/I.
Now consider $(R/I)/u$. We are setting $u = 0$, so

$$R/(I, u) \cong \mathbb{F}\begin{bmatrix} 0 & v & w \\ x & y & z \end{bmatrix}/(vz - wy, wx, vx)$$

Certainly w, v and x are zero divisors in the above ring. But the element $v - x$ is not. One can convince oneself of this fact by multiplying $v - x$ by any variable, v, w, x, y or z and getting something that is not a zero divisor.
Now consider $(R/I)/(u, v - x)$. We are setting $v - x = 0$, or in other words, $v = x$. So

$$R/(I, u, v - x) = \mathbb{F}\begin{bmatrix} 0 & v & w \\ v & y & z \end{bmatrix}/(vz - wy, v^2, wv)$$

Certainly, w and v are zero divisors in the above ring, but again, we can convince ourselves that $w - y$ is not a zero divisor.

$$R/(I, u, v - x, w - y) \cong \mathbb{F}\begin{bmatrix} 0 & v & w \\ v & w & z \end{bmatrix}/(vz - w^2, v^2, wv)$$

Finally,

$$R/(I, u, v - x, w - y, z) \cong \mathbb{F}\begin{bmatrix} 0 & v & w \\ v & w & 0 \end{bmatrix}/(w^2, v^2, wv)$$

Since every non-unit is a zerodivisor in the above ring, not only do we have a regular sequence, but we have a *maximal* regular sequence.

Definition 8.1.2. *Cohen-Macaulay*

If there is some *(equivalently, every)* system of parameters that is a regular sequence, than the ring is called Cohen-Macaulay.

Thus, the ring R/I is Cohen-Macaulay (nice!). It is not, however, Gorenstein, a definition beyond the scope of this talk (not too nice!).

8.2 Relation to Representation Theory

Notice that

$$\begin{aligned} 2 \times 3 \text{ matrices of elements of } \mathbb{C} &= \text{Hom}\left(\mathbb{C}^3, \mathbb{C}^2\right) \\ &= \left(\mathbb{C}^3\right)^* \otimes \mathbb{C}^2 \end{aligned}$$

where a matrix acts on an element of $\mathbb{C}^3$ by multiplication on the left.
Define the group $GL(3) \times GL(2)$ action on $\left(\mathbb{C}^3\right)^* \otimes \mathbb{C}^2$ in the following way. Let $\phi \in \text{Hom}(\mathbb{C}^3, \mathbb{C}^2)$. Then

$$(g_1, g_2)\phi \mapsto g_2^{-1}\phi(g_1^{-1}).$$

After fixing a basis, let $\mathbf{Sym}(V)$ denote the symmetric algebra on the basis of V.

$$\mathbf{Sym}(\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^2)) = \mathbf{Sym}(2 \times 3 \text{ matrices with elements in } \mathbb{C}$$
$$= \mathbb{C}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix} = R$$

Then we can think of R/I as a subrepresentation of R, that is, all matrices of rank 1 or less, **or** as a quotient representation of R. If M is some matrix that has rank 1 or less, then action by $GL(3) \times GL(2)$ will take this matrix to some other element of rank 1 or less.

Let E be the vector space $\mathbb{C}^3$ and F be the vector space $\mathbb{C}^2$, and let e_1, e_2, e_3 be a basis for E and f_1, f_2 be a basis for F. Then $\mathbf{Sym}^1(E^* \otimes F) \cong E^* \otimes F$ has basis:

$$\begin{bmatrix} e_1^* \otimes f_1 & e_2^* \otimes f_1 & e_3^* \otimes f_1 \\ e_1^* \otimes f_2 & e_2^* \otimes f_2 & e_3^* \otimes f_2 \end{bmatrix}$$

and degree 1 polynomials in $R = \mathbb{C}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$ have, as a vector space basis, $\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$.

We might expect symmetric algebras to play nicely with tensor products, and that $\mathbf{Sym}^2(E^* \otimes F)$ might be the same as $\mathbf{Sym}^2(E^*) \otimes \mathbf{Sym}^2(F)$. However, in $\mathbf{Sym}^2(E^* \otimes F)$,

$$(e_1^* \otimes f_1)(e_2^* \otimes f_2) \neq (e_1^* \otimes f_2)(e_2^* \otimes f_1)$$

but in $\mathbf{Sym}^2(E^*) \otimes \mathbf{Sym}^2(F)$ the analagous element for both of the above elements is:

$$e_1^* e_2^* \otimes f_1 f_2$$

In other words, in $\mathbf{Sym}^2(E^* \otimes F)$, $uy \neq vx$, but $\mathbf{Sym}^2(E^*) \otimes \mathbf{Sym}^2(F)$ acts just like R/I!

Suppose we want to understand polynomials in degree m in $\mathbb{C}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix} \cong \mathbf{Sym}^m(E^* \otimes F)$. One way to do so is to understand its decomposition into symmetric products of E^* and symmetric products of F. The Cauchy Formula tells us how to do that.

Cauchy Formula If $\mathbb{F}$ a field of characteristic 0, E, F vector spaces, then

$$\mathbf{Sym}^m(E \otimes F) = \bigoplus_{|\lambda|=m} S_\lambda E \otimes S_\lambda F$$

where $S_\lambda(V)$ is the Schur functor acting on the vector space V, that is, $S_\lambda(V) = \mathrm{Im}(c_\lambda|_{V^{\otimes d}})$, where c_λ is the Young symmetrizer.

9 Decomposing tensor products of Weyl modules

Talk by Peter McDonald, notes by Peter McDonald

Throughout this section, let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of d, let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of d and let $\nu = (\nu_1, \ldots, \nu_k)$ be a partition of $d+m$.

Recall that given λ, we can construct a Young tableau and define the following sets

$$P_\lambda = \{\sigma \in S_d : \sigma \text{ preserves the rows of the Young tableau}\}$$
$$Q_\lambda = \{\sigma \in S_d : \sigma \text{ preserves the columns of the Young tableau}\}$$

Letting

$$a_\lambda = \sum_{\sigma \in P_\lambda} e_\sigma$$
$$b_\lambda = \sum_{\sigma \in Q_\lambda} \operatorname{sgn}(\sigma) e_\sigma$$

define the Young symmetrizer

$$c_\lambda = a_\lambda \cdot b_\lambda$$

Considering the action of c_λ on $V^{\otimes d}$, we get a Weyl module which we denote

$$\mathbb{S}_\lambda V = c_\lambda(V^{\otimes d})$$

which we can use as a building block of our representations. Then Schur-Weyl duality gives us that

$$V^{\otimes d} \cong \bigoplus_{|\lambda|=d} \mathbb{S}_\lambda V^{m_\lambda}$$

where m_λ is the dimension of V_λ, the irreducible representation of S_d corresponding to λ.

Now that we have these Weyl modules, we would like to understand how the tensor product of two Weyl modules behave. Intuitively, the tensor product of two Weyl modules $\mathbb{S}_\lambda$ and $\mathbb{S}_\mu$ can be decomposed into components of $V^{\otimes d+m}$, but we want to know exactly which components their product corresponds to. In fact, given λ a partition of d and μ a partition of m, we have the following isomorphism

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \cong \bigoplus_{|\nu|=d+m} N_{\lambda\mu\nu} \mathbb{S}_\nu V$$

where $N_{\lambda\mu\nu}$ are numbers determined by the *Littlewood-Richardson* rule. While we will not prove this formula, we will investigate the Littlewood-Richardson rule and look at examples in the case where $\mu = (m)$ and $\mu = (1, \ldots, 1)$.

Definition 9.0.1. *Given an endomorphism g of V, we have an induced endomorphism g' of $\mathbb{S}_\lambda V$. Let $\chi_{\mathbb{S}_\lambda V}(g)$ denote the trace of g'. This will be a symmetric polynomial of degree d in k-variables, each representing an eigenvalue of x. This polynomial with indeterminates $x_1, \ldots, x_k$ is known as the **Schur polynomial** and is denoted S_λ.*

Remark 9.0.2. *$\{S_\lambda\}_{|\lambda|=d}$ is a basis for the symmetric polynomials of degree d.*

Given that Schur polynomials form a basis for the symmetric polynomials, we would like a systematic way to express the product of two Schur polynomials in terms of the basis for the corresponding degree of the product. It turns out that

$$S_\lambda \cdot S_\mu = \sum_\nu N_{\lambda\mu\nu} S_\nu$$

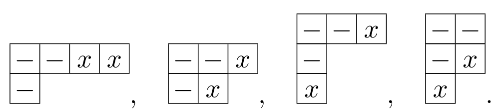
so we will need to understand how to calculate these $N_{\lambda\mu\nu}$.

Proposition 9.0.3 (Pieri's Formula)**.** *$S_\lambda \cdot S_{(m)} = \sum_\nu S_\nu$ where ν ranges over the partitions of $d+m$ whose Young diagrams are obtained from λ's by adding m boxes, no two in the same column, i.e., all $\nu = (\nu_1, \ldots, \nu_k)$ with*

$$\nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots \geq \nu_k \geq \lambda_k \geq 0$$

Example 9.0.4. $S_{(2,1)} \cdot S_{(2)} = S_{(4,1)} + S_{(3,2)} + S_{(3,1,1)} + S_{(2,2,1)}$

Proof. The corresponding Young diagrams are, where the x's denote the two boxes added:



∎

While this can be combined with the determinental formula to find the product of any two Schur polynomials, there is an easier way.

Definition 9.0.5. *Given $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of d and $\mu = (\mu_1, \ldots, \mu_k)$ a partition of m, a **μ-extension** of the Young diagram for λ is obtained by the following:*

1. *Add μ_1 boxes to λ's Young diagram according to Pieri's formula, marking the boxes with a 1*

2. *Add μ_2 boxes to the above Young diagram according to Pieri's formula, marking the boxes with a 2*

3. *Continue this process for all remaining μ_i*

*We say that a μ-expansion is **strict** if, when reading the diagram from right to left and top to bottom, at any point in the reading the integer p appears at least as any times as the integer $p+1$ for $1 \leq p \leq k-1$.*

Proposition 9.0.6 (Littlewood-Richardson Rule)**.** *Given λ a partition of d, μ a partition of m, and ν a partition of $d+m$, $N_{\lambda\nu\mu}$ is the number of ways the Young diagram for λ can be expanded to a Young diagram for ν by a μ-strict expansion.*

Recall our formula for the product of two Schur polynomials

$$S_\lambda \cdot S_\mu = \sum_\nu N_{\lambda\mu\nu} S_\nu$$

We compute an example

Example 9.0.7. $S_{(2,1)} \cdot S_{(2,1)} = S_{(4,2)} + S_{(4,1,1)} + S_{(3,3)} + 2S_{(3,2,1)} + S_{(3,1,1,1)} + S_{(2,2,2)} + S_{(2,2,1,1)}$

Proof. The $(2,1)$-extensions of $(2,1)$ are listed below:

$$\begin{array}{cccc} - & - & 1 & 1 \\ - & 2 & & \end{array}\;,\quad \begin{array}{cccc} - & - & 1 & 1 \\ - & & & \\ 2 & & & \end{array}\;,\quad \begin{array}{ccc} - & - & 1 \\ - & 1 & 2 \end{array}\;,\quad \begin{array}{ccc} - & - & 1 \\ - & 1 & \\ 2 & & \end{array}\;,$$

$$\begin{array}{ccc} - & - & 1 \\ - & 2 & \\ 1 & & \end{array}\;,\quad \begin{array}{ccc} - & - & 1 \\ - & & \\ 1 & & \\ 2 & & \end{array}\;,\quad \begin{array}{cc} - & - \\ - & 1 \\ 1 & 2 \end{array}\;,\quad \begin{array}{cc} - & - \\ - & 1 \\ 1 & \\ 2 & \end{array}$$

□

The fact that these Littlewood-Richardson coefficients come from the multiplication of Schur polynomials is no mistake. Because Schur polynomials correspond to the characters of the Weyl modules, the character of the product of two Weyl modules is the product of their characters. Now that we understand where the coefficients $N_{\lambda\mu\nu}$ come from, we can compute a few decompositions of the tensor products of Weyl-modules.

Example 9.0.8.

$$\mathbb{S}_\lambda \otimes \mathbb{S}_{(m)} = \mathbb{S}_\lambda \otimes Sym^m V \cong \bigoplus_\nu N_{\lambda\mu\nu}\mathbb{S}_\nu$$

where ν is all partitions of $d+m$ whose Young diagrams are obtained from λ's by adding m boxes, no two in the same column.

Example 9.0.9.

$$\mathbb{S}_\lambda \otimes \mathbb{S}_{(1,\ldots,1)} = \mathbb{S}_\lambda \otimes \Lambda^m V \cong \bigoplus_\nu N_{\lambda\mu\nu}\mathbb{S}_\nu$$

where ν is all partitions of $d+m$ whose Young diagrams are obtained from λ's by adding m boxes, no two in the same row.