**Optimization Practice**

Name: __________________________ Score: ______________

Math 1210 - 001 Summer 2012 Instructor: Katrina Johnson

Clearly indicate the meaning of each variable as you set up each problem, draw a diagram if necessary. Answer each part of the following questions to complete each problem. Feel free to use a calculator. Show all of your work. Don't forget units. This assignment is due: Monday, July 23. I will not accept late assignments!

(4 pts) 1. During the summer months Terry makes and sells necklaces on the beach. Last summer he sold the necklaces for $10 each and his sales averaged 20 necklaces per day. When he increased the price by $1, he found that he lost two sales per day. So the price of a necklaces is represented by the function, \( k(x) = \frac{1}{2}x + 20 \), and the revenue is modeled by the function, \( R(x) = \frac{1}{2}x^2 + 20x \), where \( x \) represents the number of necklaces sold.

The material for each necklace cost Terry $6. This is represented by the cost function, \( C(x) = 6x \), where \( x \) represents the number of necklaces produced. What should the selling price be to maximize Terry's profit? Hint: Recall the profit function is \( P(x) = R(x) - C(x) \).

i) You would like to minimize/maximize (circle one) profit.

ii) What is the objective function?

\[ P(x) = \left( \frac{1}{2}x^2 + 20x \right) - (6x) = \frac{1}{2}x^2 + 20x - 6x \]

iii) What is the derivative of the objective function?

\[ P'(x) = -x + 14 \]

iv) What are the critical values of the objective function? At which critical values do you minimize/maximize the objective function? How do you know?

\[ P'(x) = 0 \Rightarrow -x + 14 = 0 \]

\[ x = 14 \]

\[ P' \quad + \quad ^{\text{max}} \quad - \]

\[ x = 14 \]

v) What should the selling price be to maximize the profit?

\[ x = 14 \]

\[ k(14) = \frac{1}{2}(14) + 20 = -7 + 20 = 13 \]

Terry should sell the necklaces for $13 each to maximize his profit.
2. A dairy farmer plans to enclose a rectangular pasture adjacent to a river. To provide enough grass for the herd, the pasture must contain 180,000 square meters. No fencing is required along the river. What dimensions use the least amount of fencing?

i) You would like to **minimize**/maximize (circle one) amount of fencing.

ii) a) What is the objective function? \( P = 2x + y \)
   
   b) What are the constraints? \( xy = 180,000 \)

   c) Solve the constraints for one variable and substitute to get the objective function as a function in one variable.

   \[
   y = \frac{180,000}{x} \]

   \[
   P = 2x + \frac{180,000}{x} = 2x + 180,000x^{-1} \]

   \[
   \]

   \[
   iii) What is the derivative of the objective function?
   \[
   P' = 2 - 180,000x^{-2} = 2 - \frac{180,000}{x^2} = \frac{2x^2 - 180,000}{x^2} \]

   \[
   iv) What are the critical values of the objective function? At which critical values do you minimize/maximize the objective function? How do you know?
   \[
   P' = 0 \Rightarrow 2x^2 - 180,000 = 0
   \]

   \[
   x^2 = 90,000 \]

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   x = \pm 300 \]

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(4 pts) 3. A manufacturer wants to design an open box that has a square base and a surface area of 108 square inches. What dimensions will produce a box with a maximum volume?

i) You would like to minimize/maximize (circle one) __________ volume.

ii) a) What is the objective function? \( V = x^2 h \)
   b) What are the constraints? \( S.A. = 108 = 4xh + x^2 \)
   c) Solve the constraints for one variable and substitute to get the objective function as a function in one variable.

\[
108 = 4xh + x^2 \Rightarrow 108 - x^2 = 4xh \Rightarrow h = \frac{108 - x^2}{4x}
\]

\[
V = x^2 \left( \frac{108 - x^2}{4x} \right) = \frac{1}{4} x \left( 108 - x^2 \right) = -\frac{1}{4} x^3 + 27x
\]

iii) What is the derivative of the objective function?

\[
V' = -\frac{3}{4} x^2 + 27
\]

iv) What are the critical values of the objective function? At which critical values do you minimize/maximize the objective function? How do you know?

\[
V' = 0 \Rightarrow \frac{3}{4} x^2 = 27
\]

\[
x^2 = 36
\]

\[
x = \pm 6
\]

but \( x > 0 \), so \( x = 6 \)

v) What dimensions will produce a box with a maximum volume?

\[
x = 6 \quad h = \frac{108 - (6)^2}{4(6)} = \frac{108 - 36}{24} = 3
\]

\[6 \text{ in.} \times 6 \text{ in.} \times 3 \text{ in.} \]
(4 pts) 4. A soft drink can is to contain 12 fluid ounces. Assume the can is made by forming sheet aluminum into a cylindrical container (open on one end) and then attaching a circular lid. The bottom and sides cost $0.02 per square meter, and the top costs $0.05 per square meter. What is the ratio of height to base diameter of the can that is cheapest to manufacture?

\[
1 \text{ fluid ounce} = 2.957 \times 10^{-5} \text{ m}^3
\]

i) You would like to **minimize** (circle one) **cost**.

ii) a) What is the objective function? 
\[ C = 0.02 (\pi r^2 + 2\pi rh) + 0.05 (\pi \frac{h}{2}) \]
b) What are the constraints? 
\[ 3.5484 \times 10^{-4} = \pi r^2 h \]
c) Solve the constraints for one variable and substitute to get the objective function as a function in one variable.

\[
C = 0.07\pi r^2 + 1.419 \times 10^{-5} \text{ r}^{-1}
\]

iii) What is the derivative of the objective function?

\[
C' = 0.14\pi r - 1.419 \times 10^{-5} \text{ r}^{-2}
\]

iv) What are the critical values of the objective function? At which critical values do you minimize/maximize the objective function? How do you know?

\[
C' = 0 \implies 0.14\pi r^3 = 1.419 \times 10^{-5}
\]

\[
r^3 = 3.227 \times 10^{-5}
\]

\[
r = 0.0318 \text{ m}
\]

v) What is the ratio of height to base diameter of the can that is cheapest to manufacture?

\[
h = \frac{3.5485 \times 10^{-4}}{\pi (0.0318)^2} = 0.1114 \text{ m}
\]

\[
d = 2(0.0318) = 0.0636 \text{ m}
\]

\[
\text{ratio } h : d = \frac{0.1114 \text{ m}}{0.0636 \text{ m}} = 1.75
\]
(4 pts) 5. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

\[ A = 2 \left[ x \left( 8 - x^2 \right) \right] \]
\[ A = 2 \left( 8x - x^3 \right) = 16x - 2x^3 \]

\[ A' = 16 - 6x^2 \]
\[ A' = 0 \implies 16 = 6x^2 \]
\[ x^2 = \frac{8}{3} \]
\[ x = \pm \sqrt{\frac{8}{3}} = \pm \frac{2\sqrt{6}}{3} \]

but $x > 0$ so $x = \frac{2\sqrt{6}}{3}$

\[ x = \frac{2\sqrt{6}}{3} \]

\[ y = 8 - \left( \frac{2\sqrt{6}}{3} \right)^2 = 8 - \frac{8}{3} = \frac{16}{3} \]

base is $2x = 2 \left( \frac{2\sqrt{6}}{3} \right) = \frac{4\sqrt{6}}{3}$

height is $y = \frac{16}{3}$

\[ \frac{4\sqrt{6}}{3} \text{ by } \frac{16}{3} \]