Lab Project 3 (Due 4/22/03)
MATH 2270-1 - Spring 2003

From the course webpage download the file proj3data.m. Load it (this is explained in
the third Maple tutorial, also available at the course webpage) into Maple and you will have
defined a few vectors that will be used throughout this project. The naming convention is
as follows: the vector \( v_1 \) in this note becomes the vector \( vv1 \), and similarly for the other
vectors.

1. The vectors \( v_1, v_2, v_3, v_4 \in \mathbb{R}^{10} \) are loaded in Maple (after, of course, you load the file
proj3data.m). Define the subspace \( S = \langle v_1, v_2, v_3, v_4 \rangle \subseteq \mathbb{R}^{10} \).

   (a) Check if the Pythagorean relation \( |v_1|^2 + |v_2|^2 = |v_1 + v_2|^2 \) holds. What do you
       conclude from this?

   (b) Find the angle between \( v_1 \) and \( v_2 \).

   (c) Find a basis \( \mathcal{B} \) of \( S \). What is \( \dim S \)?

   (d) Follow the Gram-Schmidt procedure to transform \( \mathcal{B} \) into an orthonormal basis \( \mathcal{B}' \)
       of \( S \).

   (e) Use the command QRDecomposition of Maple’s LinearAlgebra to find an or-
      thonormal basis of \( S \).

   (f) Find the components of \( v_1 + v_2 + v_3 + v_4 \) with respect to \( \mathcal{B}' \).

   (g) Let \( P_S : \mathbb{R}^{10} \to \mathbb{R}^{10} \) be the orthogonal projection onto \( S \). Find the matrix of \( P_S \)
       with respect to the canonical basis of \( \mathbb{R}^{10} \).

   (h) Find the image of \( w \) (already defined in Maple) under \( P_S \). What is the angle
       between \( P_S(w) \) and \( w - P_S(w) \)?

   (i) Let \( R_S : \mathbb{R}^{10} \to \mathbb{R}^{10} \) be the reflection on \( S \) and \( MRS \) its matrix with respect to
       the canonical basis of \( \mathbb{R}^{10} \). Find \( MRS \).

   (j) Find \( R_S(w) \). Compare \( |w| \) and \( |R_S(w)| \). Could \( R_S \) be an orthogonal transformation?

   (k) Use \( MRS \) to decide if \( R_S \) is an orthogonal transformation or not.

2. For this exercise you may want to refer to the Maple worksheet on trigonometric
   polynomials available on the class’ webpage. Let \( V \) be the space of continuous functions
   on the interval \([-\pi, \pi]\) with the inner product \( \langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt \).

   We will use the notation \( a[n](t) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } n = 0, \\ \cos(nt) & \text{if } n > 0 \end{cases} \) and \( b[n](t) = \sin(nt) \) for \( n > 0 \).

   Clearly \( a[n] \) and \( b[n] \) are in \( V \) for all \( n \).

   (a) Use Maple to verify that all \( a[n] \) and \( b[n] \) have length 1 for all \( n \). Hint: Evaluate
       the integrals in Maple; for example, evaluate \( \int_{-\pi}^{\pi} \sin^2(nt)dt \). Do not try to define
       functions \( a[n](t) \) for arbitrary \( n \).

   (b) Use Maple to verify that all \( a[n] \) and \( b[m] \) are orthogonal. The same for \( a[n] \) and
       \( a[m] \) with \( n \neq m \) and for \( b[n] \) and \( b[m] \) with \( n \neq m \).
(c) Let \( f(x) = x^2 - 1 \). Find the orthogonal projection of \( f \) on the subspace \( T_1 = \langle a[0], a[1], b[1] \rangle \). Write explicitly \( P_{T_1}(f) \). Plot \( f \) and \( P_{T_1}(f) \) together. Compute the “error” of the approximation of \( f \) by \( P_{T_1}(f) \): \(|f - P_{T_1}(f)|\).

(d) Find the orthogonal projection of \( f \) on the subspace \( T_2 = \langle a[0], a[1], a[2], b[1], b[2] \rangle \). Write explicitly \( P_{T_2}(f) \). Plot \( f \) and \( P_{T_2}(f) \) together. Compute the “error” of the approximation of \( f \) by \( P_{T_2}(f) \): \(|f - P_{T_2}(f)|\).

(e) Find the first approximation where the error is less than 0.2.

3. In this exercise we want to study the volume of some parallelepipeds.

(a) Consider vectors \( u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Find the volume (area, in this case) of the parallelepiped spanned by \( u_1 \) and \( u_2 \).

(b) Consider a pair of vectors \( u_1, u_2 \in \mathbb{R}^2 \) whose components can only be 0, 1. For each such pair, you can compute the volume of the parallelepiped spanned by \( u_1 \) and \( u_2 \). What is the maximum volume that you can obtain by this procedure? Show a pair of vectors that achieve this maximum.

(c) Same as in the previous item but for three vectors in \( \mathbb{R}^3 \).

(d) Same as above but for 4 vectors in \( \mathbb{R}^4 \).