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Topology

Definition

metric space

Topology

Definition

subspace

Topology

Definition

isometry

Topology

Definition

open set

Topology

Proposition

open balls are open

Topology

Theorem

unions and intersections of open sets

Topology

Definition

closed set

Topology

Definition

closed ball

Topology

Proposition

closed balls are closed sets

Topology

A **metric space** (X, d) is a set X and a function $d : X \times X \to \mathbb{R}$ satisfying $\forall\, x, y, z \in X$

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

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Suppose (X_1, d_1) and (X_2, d_2) are metric spaces. A function $f : X_1 \to X_2$ is called an **isometry** if f is one–to–one, onto and

$$d_2(f(x), f(y)) = d_1(x, y) \quad \forall\, x, y \in X_1$$

If (X, d) is a metric space, and $A \subset X$ then $(A, d|_{A\times A})$ is a metric space and is called a **subspace** of (X, d).

If (X, d) is a metric space, then for each $x \in X$ and for each $r > 0$, $B(x, r)$ is open in X.

Supposing (X, d) is a metric space, then a subset $U \subset X$ is **open** iff

$$\forall\, x \in U, \exists\, r > 0 \text{ such that } B(x, r) \subset U$$

Let (X, d) be a metric space, $F \subset X$ is **closed** iff $X - F$ is open.

Let (X, d) be a metric space and let $\{U_\alpha\}_{\alpha\in A}$ be any collection of open sets in (X, d), then

1. $X, \varnothing$ are open.
2. $\bigcup_{\alpha\in A} U_\alpha$ is open.
3. Let $\{U_1, \ldots, U_n\}$ be a finite collection of open sets, then $\bigcap_{i=1}^{n} U_i$ is open.

A closed ball $\overline{B}(x, r)$, is a closed set.

A **closed ball** centered at x of radius r is denoted $\overline{B}(x, r)$, and defined to be:

$$\overline{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}$$

Theorem

unions and intersections of closed sets

Topology

Definition

closure

Topology

Definition

distance from a point to a set

Topology

Definition

Cauchy Sequence

Topology

Theorem

convergence implies Cauchy

Topology

Definition

interior

Topology

Definition

exterior & frontier

Topology

Definition

limit of a sequence

Topology

Definition

convergent sequence

Topology

Definition

complete metric space

Topology

Let (X, d) be a metric space with $A \subset X$. The **interior** of A denoted A° is defined to be:

$$A^\circ = \{x \in A \mid \exists\, r > 0 \text{ such that } B(x, r) \subset A\}$$

Let (X, d) be a metric space and let $\{F_\alpha\}_{\alpha \in A}$ be any collection of closed sets in (X, d), then

1. $X, \varnothing$ are closed.
2. $\bigcap_{\alpha \in A} F_\alpha$ is closed.
3. Let $\{F_1, \ldots, F_n\}$ be a finite collection of closed sets, then $\bigcup_{i=1}^{n} F_i$ is closed.

Let (X, d) be a metric space with $A \subset X$.

The **exterior** of a set A is defined to be $(X - A)^\circ$.

The **frontier** of a set A is defined to be $\overline{A} - A^\circ$.

Let (X, d) be a metric space with $A \subset X$. The **closure** of A denoted $\overline{A}$ is defined to be:

$$\overline{A} = \{x \in X \mid \forall\, r > 0, B(x, r) \cap A \neq \varnothing\}$$

Suppose (X, d) is a metric space. A sequence $\{x_n\} \subset X$ has **limit** x, denoted $\lim_{n \to \infty} \{x_n\} = x$ iff

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that}$$

$$n \geq N \Rightarrow x_n \in B(x, \varepsilon)$$

Suppose (X, d) is a metric space with $A \subset X$ and $x \in X$. We define **the distance from x to A** by

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}$$

A sequence $\{x_n\}$ **converges** iff $\lim \{x_n\}$ exits.

Suppose (X, d) is a metric space. A sequence $\{x_n\} \subset X$ is called a **Cauchy sequence** iff

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that}$$

$$m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon$$

A metric space (X, d) is **complete** iff every Cauchy sequence in X is convergent.

If a sequence $\{x_n\}$ is convergent then it is Cauchy.

Theorem

limits are unique

Topology

Definition

continuous function

Topology

Definition

Lipschitz function

Topology

Definition

bi-Lipschitz

Topology

Theorem

continuous functions and sequences

Topology

Theorem

distinct points have a radius of separation

Topology

Definition

continuous function (alternate definition)

Topology

Theorem

Lipschitz functions are uniformly continuous

Topology

Theorem

f *continuous iff*
the preimage of every open set is open

Topology

Definition

homeomorphism

Topology

Suppose (X, d) is a metric space, and $x, y \in X$ with $x \neq y$, then $\exists\, r > 0$ such that $B(x, r) \cap B(y, r) = \varnothing$

If the limit of $\{x_n\}$ exists, then that limit is unique.

Suppose $(X_1, d_1), (X_2, d_2)$ are metric spaces. A function $f : X_1 \to X_2$ is **continuous** on X_1 iff

$$\forall\, x \in X_1,\ \forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that}$$

$$f(B(x, \delta)) \subset B(f(x), \varepsilon)$$

Suppose $(X_1, d_1), (X_2, d_2)$ are metric spaces. A function $f : X_1 \to X_2$ is **continuous** at $x \in X_1$ iff

$$\forall\, \varepsilon > 0,\ \exists\, \delta(x, \varepsilon) > 0 \text{ such that}$$

$$d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$$

If $f : X_1 \to X_2$ is Lipschitz on X_1, then f is uniformly continuous on X_1.

Suppose $(X_1, d_1), (X_2, d_2)$ are metric spaces. A function $f : X_1 \to X_2$ is called **Lipschitz** iff

$$\forall\, x, y \in X_1\ \exists\, c > 0 \text{ such that}$$

$$d_2(f(x), f(y)) \leq c d_1(x, y)$$

A Lipschitz function can be thought of as a "bounded distortion."

A function $f : X_1 \to X_2$ is continuous iff

$$\forall\, U \text{ open } \subset X_2 \Rightarrow f^{-1}(U) \text{ open } \subset X_1$$

Or equivalently:

$$\forall\, U \text{ closed } \subset X_2 \Rightarrow f^{-1}(U) \text{ closed } \subset X_1$$

Suppose $(X_1, d_1), (X_2, d_2)$ are metric spaces. A function $f : X_1 \to X_2$ is called **bi-Lipschitz** iff

$$\forall\, x, y \in X_1\ \exists\, c_1, c_2 > 0 \text{ such that}$$

$$c_1 d_1(x, y) \leq d_2(f(x), f(y)) \leq c_2 d_1(x, y)$$

A function $f : (X_1, d_1) \to (X_2, d_2)$ is called a **homeomorphism** iff

1. f is continuous
2. f is 1-1 and onto
3. f^{-1} is continous

A function $f : (X_1, d_1) \to (X_2, d_2)$ is continuous iff

$$\forall \text{ convergent sequences } \{x_n\} \subset X_1,$$

$$\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} \{x_n\})$$

Definition

equivalent metrics

Topology

Remark

two metrics are equivalent iff the identity map is a homeomorphism

Topology

Theorem

composition of continuous functions preserves continuity

Topology

Definition

homeomorphic spaces

Topology

Definition

topology

Topology

Definition

topological space

Topology

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Topology

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Topology

Two metrics, d_1, d_2 are equivalent iff $id : (X, d_1) \to (X, d_2)$ is a homeomorphism.

Two metrics d_1, d_2 are called **equivalent** iff they have the same open sets.

Two metric spaces are **homeomorphic** iff there exists a homeomorphism between them.

Suppose $f : X_1 \to X_2$ and $g : X_2 \to X_3$. If f and g are continuous then $g \circ f$ is continuous.

A **topological space** (X, τ) is a set X and a topology τ on X.

Suppose X is a set. A collection τ of subsets of X is called a **topology** on X iff

1. $X \in \tau$ and $\varnothing \in \tau$
2. $U_\alpha \in \tau$ for $\alpha \in A \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \tau$
3. $U_1, U_2, \ldots, U_n \in \tau \Rightarrow \bigcap_{i=1}^{\infty} U_i \in \tau$