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reduced row–echelon form

Linear Algebra

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rank

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Definition & Theorem

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number of vectors in a basis

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constructing a basis of the image

Linear Algebra

The *rank* of a matrix A, is the number of leading 1s in $rref(A)$.

A matrix is in *reduced row–echelon form* if all of the following conditions are satisfied:

1. If a row has nonzero entries, then the first nonzero entry is 1.
2. If a column contains a leading 1, then all other entries in that column are zero.
3. If a row contains a leading 1, then each row above contains a leading 1 further to the left.

A *linear combination* is a vector in $\mathbb{R}^n$ created by adding together scalar multiples of other vectors in $\mathbb{R}^n$. For example, if $c_1, \ldots, c_m$ are in $\mathbb{R}$, then

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

is a *linear combination*. We say $\vec{x}$ is a *linear combination* of $\vec{v}_1, \ldots, \vec{v}_m$.

If a system has at least one solution, then it is said to be *consistent*. If a system has no solutions, then it is said to be *inconsistent* (*overdetermined*).

A *consistent* system has either

- infinitely many solutions (*underdeterminied*)
- exactly one solution (*exactly determined*)

If $T(\vec{x}) = A\vec{x}$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

- $ker(T) = ker(A)$ is a subspace of $\mathbb{R}^m$
- $im(T) = im(A)$ is a subspace of $\mathbb{R}^n$

A subset W of $\mathbb{R}^n$ is called a *subspace* of $\mathbb{R}^n$ if it has the following three properties:

1. W contains the zero vector for $\mathbb{R}^n$.
2. W is closed under addition (if $\vec{w}_1$ and $\vec{w}_2$ are both in W, then so is $\vec{w}_1 + \vec{w}_2$).
3. W is closed under scalar multiplication (if $\vec{w}$ is in W and k is any scalar, then $k\vec{w}$ is also in W).

Consider vectors $\vec{v}_1, \ldots, \vec{v}_m$ from a subspace V of $\mathbb{R}^n$. These vectors are said to form a *basis* of V, if they meet the following two requirements:

1. they span V
2. they are linearly independent

Consider vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$. These vectors are said to be *linearly independent* if none of them is a linear combination of the preceding vectors. One way to think of this is that none of the vectors is redundant.

Otherwise the vectors are said to be *linearly dependent*.

To construct a basis of the image of A, pick those column vectors of A that correspond to the columns of $rref(A)$ that contain leading 1s.

All bases of a subspace V of $\mathbb{R}^n$ consist of the same number of vectors. In other words, they all have the same dimension.

Definition

linear relations

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Definition

dimension

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Theorem

linearity

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Theorem

$ker(A) =$

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Theorem

$(AB)^T =$

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Theorem

rank-nullity theorem

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Definition

linear transformation

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Theorem

$det(A^T) =$

Linear Algebra

Theorem

$(imA)^\perp =$

Linear Algebra

Definition

symmetric and
skew-symmetric matrices

Linear Algebra

For any $n \times m$ matrix A the following equation holds:

$$dim(im(A)) + dim(ker(A)) = m$$

Alternatively, if we define the $dim(ker(A))$ to be the *nullity* of A, then we can rewrite the above as:

$$rank(A) + nullity(A) = m$$

Some mathemeticians refer to this as the fundamental theorem of linear algebra.

Consider vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$. An equation of the form

$$c_1\vec{v}_1, + \ldots + c_m\vec{v}_m = \vec{0}$$

is called a *linear relation* among the vectors $\vec{v}_1, \ldots, \vec{v}_m$. The *trivial relation* with $c_1, \ldots, c_m = 0$ is always true. *Non-trivial relations* (where at least one of the coefficients c_i is nonzero) may or may not exist among the vectors.

A function T that maps vectors from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a *linear transformation* if there is an $n \times m$ matrix A such that

$$T(\vec{x}) = A\vec{x}$$

for all $\vec{x}$ in $\mathbb{R}^m$.

For any subspace V of $\mathbb{R}^n$, the number of vectors in a basis of V is called the *dimension* of V and is denoted by $dim(V)$.

$$det(A^T) = det(A)$$

A transformation T is *linear* iff (if and only if), for all vectors $\vec{v}, \vec{w}$ and all scalars k

- $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$
- $T(k\vec{v}) = kT(\vec{v})$

$$(imA)^{\perp} = ker(A^T)$$

$$ker(A) = ker(A^TA)$$

Square matrix A is *symmetric* $\Leftrightarrow A^T = A$

Square matrix A is *skew-symmetric* $\Leftrightarrow A^T = -A$

$$(AB)^T = B^TA^T$$

Definition

orthogonal complement

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determinants of similar matrices

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the determinant in terms of the columns

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eigenvectors and eigenvalues

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properties of orthogonal matrices

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determinant of an inverse

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Theorem

elementary row operations and determinants

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Theorem

eigenvalues and characteristic equation

Linear Algebra

Theorem

characteristic equation of a 2×2 matrix

Linear Algebra

Consider an $n \times n$ matrix A. The following statements are equivalent:

1. A is an orthogonal matrix
2. The columns of A form an orthonormal basis of $\mathbb{R}^n$
3. $A^T A = I_n$
4. $A^{-1} = A^T$
5. $\forall \vec{x} \in \mathbf{R}^n \quad \|A\vec{x}\| = \|\vec{x}\| \quad$ (preserves length)

Consider a subspace V of $\mathbb{R}^n$. The *orthogonal complement* $V^\perp$ of V is the set of those vectors $\vec{x}$ in $\mathbb{R}^n$ that are orthogonal to all vectors in V:

$$V^\perp = \{\vec{x} : \vec{v} \cdot \vec{x} = 0, \forall \vec{v} \in V\}$$

Note that $V^\perp$ is the kernel of the orthogonal projection onto V.

$$det(A^{-1}) = (detA)^{-1} = \frac{1}{det(A)}$$

$$A \sim B \Rightarrow det(A) = det(B)$$

For any $n \times n$ matrix A, and any scalar k:

Elementary row op	Effect on determinant
scalar multiplication	$det(A) \to k \cdot det(A)$
row swap	$det(A) \to -det(A)$
multiple of one row added to another	$det(A) \to det(A)$

Analogous results hold for column operations.

If A is an $n \times n$ matrix with columns, $\vec{v}_1, \ldots \vec{v}_n$, then,

$$|det(A)| = \|\vec{v}_1\| \|\vec{v}_2^\perp\| \cdots \|\vec{v}_n^\perp\|$$

where $\vec{v}_1^\perp, \ldots \vec{v}_n^\perp$ are defined as in the Gram-Schmidt process.

The eigenvalues of an $n\times n$ matrix A correspond to the solutions of the *characteristic equation* given by:

$$|A - \lambda I| = 0$$

Consider an $n\times n$ matrix A. A **nonzero** vector $\vec{v}$ in $\mathbb{R}^n$ is called an *eigenvector* of A if $A\vec{v}$ is a scalar multiple of $\vec{v}$. That is, if

$$A\vec{v} = \lambda\vec{v}$$

for some scalar λ. Note that this scalar may be zero. This scalar λ is caled the *eigenvalue* associated with the eigenvector $\vec{v}$.

Given a 2×2 matrix A:

$$det(A - \lambda I) = \lambda^2 - tr(A)\lambda + det(A) = 0$$

The sum of the diagonal entries of a square matrix A is called the *trace* of A, and is denoted by $tr(A)$.