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REAL ANALYSIS I

THEOREM

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Theorem 9

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Theorem 1. *Let f be a continuous function. If $\int_0^1 f(x)\,dx \neq 0$, then there exists a point x in the interval $[0,1]$ such that $f(x) \neq 0$.*

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Theorem 3. *Let x be a real number. If $x > 0$, then $\frac{1}{x} > 0$.*

Theorem 2. *Let x be a real number. If $x > 0$, then $\frac{1}{x} > 0$.*

Theorem 5. *Let A and B be subsets of a universal set U. Then $A \cap (U \smallsetminus B) = A \smallsetminus B$.*

Theorem 4. *Let A be a set. Then $\emptyset \subseteq A$.*

Theorem 7. *If A and B are subsets of a set U and A^c and B^c are their complements in U, then*

1. $(A \cup B)^c = A^c \cap B^c$.
2. $(A \cap B)^c = A^c \cup B^c$.

Theorem 6. *Let A, B, and C be subsets of a universal set U. Then the following statements are true.*

1. $A \cup (U \smallsetminus A) = U$.
2. $A \cap (U \smallsetminus A) = \emptyset$.
3. $U \smallsetminus (U \smallsetminus A) = A$.
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
6. $A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$.
7. $A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$.

Theorem 9. *Let R be an equivalence relation on a set S. Then $\{E_x : x \in S\}$ is a partition of S. The relation "belongs to the same piece as" is the same as R. Conversely, if $\mathcal{T}$ is a partition of S, let R be defined by xRy iff x and y are in the same piece of the partition. Then R is an equivalence relation and the corresponding partition into equivalence classes is the same as $\mathcal{T}$.*

Theorem 8. *$(a,b) = (c,d)$ iff $a = c$ and $b = d$.*

Theorem *Theorem 10 (part 1)* Real Analysis I	Theorem *Theorem 10 (part 2)* Real Analysis I
Theorem *Theorem 11* Real Analysis I	Theorem *Theorem 12* Real Analysis I
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Theorem 10. *Suppose that $f : A \to B$. Let C, C_1 and C_2 be subsets of A and let D, D_1 and D_2 be subsets of B. Then the following hold:*

6. $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.
7. $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$.
8. $f^{-1}(B \setminus D) = A \setminus f^{-1}(D)$.
9. $f^{-1}(D_1 \setminus D_2) = f^{-1}(D_1) \setminus f^{-1}(D_2)$ *if* $D_2 \subseteq D_1$.

Theorem 10. *Suppose that $f : A \to B$. Let C, C_1 and C_2 be subsets of A and let D, D_1 and D_2 be subsets of B. Then the following hold:*

1. $C \subseteq f^{-1}[f(C)]$.
2. $f[f^{-1}(D)] \subseteq D$.
3. $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.
4. $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.
5. $f(C_1) \setminus f(C_2) \subseteq f(C_1 \setminus C_2)$ *if* $C_2 \subseteq C_1$.

Theorem 12. *Let $f : A \to B$ and $g : B \to C$. Then*

1. *If f and g are surjective, then $g \circ f$ is surjective.*
2. *If f and g are injective, then $g \circ f$ is injective.*
3. *If f and g are bijective, then $g \circ f$ is bijective.*

Theorem 11. *Suppose that $f : A \to B$. Let C, C_1 and C_2 be subsets of A and let D be a subset of B. Then the following hold:*

1. *If f is injective, then $f^{-1}[f(C)] = C$.*
2. *If f is surjective, then $f[f^{-1}(D)] = D$.*
3. *If f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.*

Theorem 14. *Let $f : A \to B$ and $g : B \to C$ be bijective. The the composition $g \circ f : A \to C$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Theorem 13. *Let $f : A \to B$ be bijective. Then*

1. $f^{-1} : B \to A$ *is bijective.*
2. $f^{-1} \circ f = i_A$ *and* $f \circ f^{-1} = i_B$.

Theorem 16. *Let S be a nonempty set. The following three conditions are equivalent:*

1. *S is countable.*
2. *There exists an injection $f : S \to \mathbb{N}$.*
3. *There exists a surjection $f : \mathbb{N} \to S$.*

Theorem 15. *Let S be a countable set and let $T \subseteq S$. Then T is countable.*

Theorem 18. *Let S, T and U be sets.*

1. *If $S \subseteq T$, then $|S| \leq |T|$.*
2. $|S| \leq |S|$.
3. *If $|S| \leq |T|$ and $|T| \leq |U|$, then $|S| \leq |U|$.*
4. *If $m, n \in \mathbb{N}$ and $m \leq n$, then $|\{1, 2, \ldots, m\}| \leq |\{1, 2, \ldots, n\}|$.*
5. *If S is finite, then $S < \aleph_0$.*

Theorem 17. *The set $\mathbb{R}$ of real numbers is uncountable.*

THEOREM *Theorem 19* REAL ANALYSIS I	THEOREM *Theorem 20* *Principle of Mathematical Induction* REAL ANALYSIS I
THEOREM *Theorem 21* REAL ANALYSIS I	THEOREM *Theorem 22* REAL ANALYSIS I
THEOREM *Theorem 23* *The Binomial Formula* REAL ANALYSIS I	THEOREM *Theorem 24* REAL ANALYSIS I
THEOREM *Theorem 25* REAL ANALYSIS I	THEOREM *Theorem 26* REAL ANALYSIS I
THEOREM *Theorem 27* REAL ANALYSIS I	THEOREM *Theorem 28* REAL ANALYSIS I

Theorem 20. *(Principle of Mathematical Induction) Let $P(n)$ be a statement that is either true or false for each $n \in \mathbb{N}$. Then $P(n)$ is true for all $n \in \mathbb{N}$ provided that*

1. *$P(1)$ is true, and*
2. *for each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.*

Theorem 19. *For any set S, we have $|S| < |\mathcal{P}(\mathcal{S})|$.*

Theorem 22. *$7^n - 4^n$ is a multiple of 3 for all $n \in \mathbb{N}$.*

Theorem 21. *$1+2+3+\cdots+n = \frac{1}{2}n(n+1)$ for every natural number n.*

Theorem 24. *Let $m \in \mathbb{N}$ and let $P(n)$ be a statement that is either true or false for each $n \geq m$. Then $P(n)$ is true for all $n \geq m$ provided that*

1. *$P(m)$ is true, and*
2. *for each $k \geq m$, if $P(k)$ is true, then $P(k+1)$ is true.*

Theorem 23. *(The Binomial Formula) If x and y are real numbers and $n \in \mathbb{N}$, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Theorem 26. *Let $x, y \in \mathbb{R}$ such that $x \leq y + \epsilon$ for every $\epsilon > 0$. Then $x \leq y$.*

Theorem 25. *Let x, y, and z be real numbers.*

1. *If $x + z = y + z$, then $x = y$.*
2. *$x \cdot 0 = 0$.*
3. *$(-1) \cdot x = -x$.*
4. *$xy = 0$ iff $x = 0$ or $y = 0$.*
5. *$x < y$ iff $-y < -x$.*
6. *If $x < y$ and $z < 0$, then $xz > yz$.*

Theorem 28. *Let $m, n, p \in \mathbb{Z}$. If p is a prime number and p divides the product mn, then p divides m or p divides n.*

Theorem 27. *Let $x, y \in \mathbb{R}$ and let $a \geq 0$. Then*

1. *$|x| \geq 0$.*
2. *$|x| \leq a$ iff $-a \leq x \leq a$.*
3. *$|xy| = |x| \cdot |y|$.*
4. *$|x + y| \leq |x| + |y|$. (The triangle inequality)*

THEOREM *Theorem 29* REAL ANALYSIS I	THEOREM *Theorem 30* REAL ANALYSIS I
THEOREM *Theorem 31* REAL ANALYSIS I	THEOREM *Theorem 32* REAL ANALYSIS I
THEOREM *Theorem 33* REAL ANALYSIS I	THEOREM *Theorem 34* REAL ANALYSIS I
THEOREM *Theorem 35* REAL ANALYSIS I	THEOREM *Theorem 36* *Archimedean Property of* $\mathbb{R}$ REAL ANALYSIS I
THEOREM *Theorem 37* REAL ANALYSIS I	THEOREM *Theorem 38* REAL ANALYSIS I

Theorem 30. *Every non-empty subset of* $\mathbb{R}$ *that is bounded below has a greatest lower bound.*

Theorem 29. *Let* p *be a prime number. Then* $\sqrt{p}$ *is not a rational number.*

Theorem 32. *Let* A *and* B *be non-empty subsets of* $\mathbb{R}$. *Then*

1. $\inf A \leq \sup A$.
2. $\sup(-A) = -\inf A$ *and* $\inf(-A) = -\sup A$.
3. $\sup(A+B) = \sup(A) + \sup(B)$ *and* $\inf(A+B) = \inf(A) + \inf(B)$.
4. $\sup(A-B) = \sup(A) - \inf(B)$.
5. *If* $A \subseteq B$, *then* $\sup A \leq \sup B$ *and* $\inf B \leq \inf A$.

Theorem 31. *Let* A *be a non-empty subset of* $\mathbb{R}$ *and* x *an element of* $\mathbb{R}$. *Then*

1. $\sup A \leq x$ *iff* $a \leq x$ *for every* $a \in A$.
2. $x < \sup A$ *iff* $x < a$ *for some* $a \in A$.

Theorem 34. *Let* f *and* g *be functions defined on a set containing* A *as a subset, and let* $c \in \mathbb{R}$ *be a positive constant. Then*

1. $\sup_A cf = c \sup_A f$ *and* $\inf_A cf = c \inf_A f$.
2. $\sup_A(-f) = -\inf_A f$.
3. $\sup_A(f+g) \leq \sup_A f + \sup_A g$ *and* $\inf_A f + inf_A g \leq \inf_A(f+g)$.
4. $\sup\{f(x) - f(y) : x, y \in A\} \leq \sup_A f - \inf_A f$.

Theorem 33. *Suppose that* D *is a nonempty set and that* $f : D \to \mathbb{R}$ *and* $g : D \to \mathbb{R}$. *If for every* $x, y \in D$, $f(x) \leq g(y)$, *then* $f(D)$ *is bounded above and* $g(D)$ *is bounded below. Furthermore,* $\sup f(D) \leq \sup g(D)$.

Theorem 36. *(Archimedean Property of* $\mathbb{R}$*) The set* $\mathbb{N}$ *of natural numbers is unbounded above in* $\mathbb{R}$.

Theorem 35. *The real number system* $\mathbb{R}$ *is a complete ordered field.*

Theorem 38. *Let* p *be a prime number. Then there exists a positive real number* x *such that* $x^2 = p$.

Theorem 37. *Each of the following is equivalent to the Archimedean property.*

1. *For each* $z \in \mathbb{R}$, *there exists* $n \in \mathbb{N}$ *such that* $n > z$.
2. *For each* $x > 0$ *and for each* $y \in \mathbb{R}$, *there exists* $n \in \mathbb{N}$ *such that* $nx > y$.
3. *For each* $x > 0$, *there exists* $n \in \mathbb{N}$ *such that* $0 < \frac{1}{n} < x$.

THEOREM *Theorem 39* REAL ANALYSIS I	THEOREM *Theorem 40* REAL ANALYSIS I
THEOREM *Theorem 41* REAL ANALYSIS I	THEOREM *Theorem 42* REAL ANALYSIS I
COROLLARY *Corollary 1* REAL ANALYSIS I	THEOREM *Theorem 43* REAL ANALYSIS I
LEMMA *Lemma 1* REAL ANALYSIS I	THEOREM *Theorem 44* *Heine–Borel Theorem* REAL ANALYSIS I
THEOREM *Theorem 45* *Bolzano–Weierstrass Theorem* REAL ANALYSIS I	THEOREM *Theorem 46* REAL ANALYSIS I

Theorem 40. *If x and y are real numbers with $x < y$, then there exists an irrational number w such that $x < w < y$.*

Theorem 39. *(Density of $\mathbb{Q}$ in $\mathbb{R}$) If x and y are real numbers with $x < y$, then there exists a rational number r such that $x < r < y$.*

Theorem 42.

1. *The union of any collection of open sets is an open set.*
2. *The intersection of any finite collection of open sets is an open set.*

Theorem 41.

1. *A set S is open iff $S = \text{int } S$. Equivalently, S is open iff every point in S is an interior point of S.*
2. *A set S is closed iff its complement $\mathbb{R} \setminus S$ is open.*

Theorem 43. *Let S be a subset of $\mathbb{R}$. Then*

1. *S is closed iff S contains all of its accumulation points.*
2. *cl S is a closed set.*
3. *S is closed iff $S = \text{cl } S$.*

Corollary 1.

1. *The intersection of any collection of closed sets is closed.*
2. *The union of any finite collection of closed sets is closed.*

Theorem 44. *(Heine–Borel) A subset S of $\mathbb{R}$ is compact iff S is closed and bounded.*

Lemma 1. *If S is a nonempty closed bounded subset of $\mathbb{R}$, then S has a maximum and a minimum.*

Theorem 46. *Let $\mathscr{F} = \{K_\alpha : \alpha \in \mathscr{A}\}$ be a family of compact subsets of $\mathbb{R}$. Suppose that the intersection of any finite subfamily of $\mathscr{F}$ is nonempty. Then $\bigcap\{K_\alpha : \alpha \in \mathscr{A}\} \neq \varnothing$.*

Theorem 45. *(Bolzano–Weierstrass) If a bounded subset S of $\mathbb{R}$ contains infinitely many points, then there exists at least one point in $\mathbb{R}$ that is an accumulation point of S.*

Corollary *Corollary 2* *Nested Intervals Theorem* Real Analysis I	Theorem *Theorem 47* Real Analysis I
Theorem *Theorem 48* Real Analysis I	Theorem *Theorem 49* Real Analysis I
Theorem *Theorem 50* Real Analysis I	Theorem *Theorem 51* Real Analysis I
Theorem *Theorem 52* *The Squeeze Principle* Real Analysis I	Theorem *Theorem 53* Real Analysis I
Theorem *Theorem 54* Real Analysis I	Corollary *Corollary 3* Real Analysis I

Theorem 47. *Let (s_n) and (a_n) be sequences of real numbers and let $s \in \mathbb{R}$. If for some $k > 0$ and some $m \in \mathbb{N}$, we have*

$$|s_n - s| \leq k|a_n|, \text{ for all } n > m,$$

and if $\lim a_n = 0$, then it follows that $\lim s_n = s$.

Corollary 2. *(Nested Intervals Theorem) Let $\mathscr{F} = \{A_n : n \in \mathbb{N}\}$ be a family of closed bounded intervals in $\mathbb{R}$ such that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} A_n \neq \varnothing$.*

Theorem 49. *If a sequence converges, its limit is unique.*

Theorem 48. *Every convergent sequence is bounded.*

Theorem 51. *Let (s_n) be a sequence of real numbers such that $\lim s_n = 0$, and let (t_n) be a bounded sequence. Then $\lim s_n t_n = 0$.*

Theorem 50. *A sequence (s_n) converges to s iff for each $\epsilon > 0$, there are only finitely many n for which $|s_n - s| \geq \epsilon$.*

Theorem 53. *Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. Then*

1. $\lim(s_n + t_n) = s + t$.
2. $\lim(ks_n) = ks$ *and* $\lim(k + s_n) = k + s$ *for any* $k \in \mathbb{R}$.
3. $\lim(s_n t_n) = st$.
4. $\lim\left(\frac{s_n}{t_n}\right) = \frac{s}{t}$, *provided that* $t_n \neq 0$ *for all* n *and* $t \neq 0$.

Theorem 52. *(The Squeeze Principle) If (a_n), (b_n), and (c_n) are sequences for which there is a number K such that $b_n \leq a_n \leq c_n$ for all $n > K$, and if $b_n \to a$ and $c_n \to a$, then $a_n \to a$.*

Corollary 3. *If (t_n) converges to t and $t_n \geq 0$ for all $n \in \mathbb{N}$, then $t \geq 0$.*

Theorem 54. *Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. If $s_n \leq t_n$ for all $n \in \mathbb{N}$, then $s \leq t$.*

THEOREM *Theorem 55* *Ratio Test* REAL ANALYSIS I	THEOREM *Theorem 56* REAL ANALYSIS I
THEOREM *Theorem 57* REAL ANALYSIS I	THEOREM *Theorem 58* *Monotone Convergence Theorem* REAL ANALYSIS I
THEOREM *Theorem 59* REAL ANALYSIS I	LEMMA *Lemma 2* REAL ANALYSIS I
LEMMA *Lemma 3* REAL ANALYSIS I	THEOREM *Theorem 60* *Cauchy Convergence Criterion* REAL ANALYSIS I
THEOREM *Theorem 61* REAL ANALYSIS I	THEOREM *Theorem 62* *Bolzano–Weierstrass Theorem For Sequences* REAL ANALYSIS I

Theorem 56. *Suppose that (s_n) and (t_n) are sequences such that $s_n \leq t_n$ for all $n \in \mathbb{N}$.*

1. *If $\lim s_n = +\infty$, then $\lim t_n = +\infty$.*
2. *If $\lim t_n = -\infty$, then $\lim s_n = -\infty$.*

Theorem 55. *(Ratio Test) Suppose that (s_n) is a sequence of positive terms and that the limit $L = \lim \left(\frac{s_{n+1}}{s_n} \right)$ exists. If $L < 1$, then $\lim s_n = 0$.*

Theorem 58. *(Monotone Convergence Theorem) A monotone sequence is convergent iff it is bounded.*

Theorem 57. *Let (s_n) be a sequence of positive numbers. Then $\lim s_n = +\infty$ iff $\lim \left(\frac{1}{s_n} \right) = 0$.*

Lemma 2. *Every convergent sequence is a Cauchy sequence.*

Theorem 59.

1. *If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.*
2. *If (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.*

Theorem 60. *(Cauchy Convergence Criterion) A sequence of real numbers is convergent iff it is a Cauchy sequence.*

Lemma 3. *Every Cauchy sequence is bounded.*

Theorem 62. *(Bolzano–Weierstrass Theorem For Sequences) Every bounded sequence has a convergent subsequence.*

Theorem 61. *If a sequence (s_n) converges to a real number s, then every subsequence of (s_n) also converges to s.*

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THEOREM *Theorem 65* REAL ANALYSIS I	THEOREM *Theorem 66* REAL ANALYSIS I
COROLLARY *Corollary 4* REAL ANALYSIS I	THEOREM *Theorem 67* REAL ANALYSIS I
THEOREM *Theorem 68* REAL ANALYSIS I	THEOREM *Theorem 69* REAL ANALYSIS I
THEOREM *Theorem 70* REAL ANALYSIS I	THEOREM *Theorem 71* REAL ANALYSIS I

Theorem 64. *Let (s_n) be a sequence and suppose that $m = \lim s_n$ is a real number. Then the following properties hold:*

1. *For every $\epsilon > 0$ there exists N such that $n > N$ implies that $s_n < m + \epsilon$.*
2. *For every $\epsilon > 0$ and for every $i \in \mathbb{N}$, there exists an integer $k > i$ such that $s_k > m - \epsilon$.*

Theorem 63. *Every unbounded sequence contains a monotone subsequence that has either $+\infty$ or $-\infty$ as a limit.*

Theorem 66. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x \to c} f(x) = L$ iff for every sequence (s_n) in D that converges to c with $s_n \neq c$ for all n, the sequence $(f(s_n))$ converges to L.*

Theorem 65. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x \to c} f(x) = L$ iff for each neighborhood V of L there exists a deleted neighborhood U of c such that $f(U \cap D) \subseteq V$.*

Theorem 67. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then the following are equivalent:*

(a) f does not have a limit at c.

(b) There exists a sequence (s_n) in D with each $s_n \neq c$ such that (s_n) converges to c, but $(f(s_n))$ is not convergent in $\mathbb{R}$.

Corollary 4. *If $f : D \to \mathbb{R}$ and if c is an accumulation point of D, then f can have only one limit at c.*

Theorem 69. *Let $f : D \to \mathbb{R}$ and let $c \in D$. Then the following three conditions are equivalent:*

(a) f is continuous at c.

(b) If (x_n) is any sequence in D such that (x_n) converges to c, then $\lim f(x_n) = f(c)$.

(c) For every neighborhood V of $f(c)$ there exists a neighborhood U of c such that $f(U \cap D) \subseteq V$.

Furthermore, if c is an accumulation point of D, then the above are all equivalent to

(d) f has a limit at c and $\lim_{x \to c} f(x) = f(c)$.

Theorem 68. *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, and let c be an accumulation point of D. If $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$, and $k \in \mathbb{R}$, then $\lim_{x \to c}(f+g)(x) = L + M$, $\lim_{x \to c}(fg)(x) = LM$, and $\lim_{x \to c}(kf)(x) = kL$.*

Theorem 71. *Let f and g be functions from D to $\mathbb{R}$, and let $c \in D$. Suppose that f and g are continuous at c. Then*

(a) $f + g$ and fg are continuous at c,

(b) f/g is continuous at c if $g(c) \neq 0$.

Theorem 70. *Let $f : D \to \mathbb{R}$ and let $c \in D$. Then f is discontinuous at c iff there exists a sequence (x_n) in D such that (x_n) converges to c but the sequence $(f(x_n))$ does not converge to $f(c)$.*

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Theorem 73

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COROLLARY

Corollary 5

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LEMMA

Lemma 4

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Theorem 74
Intermediate Value Theorem

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Theorem 75

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Theorem 76

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Theorem 77

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Theorem 79

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Theorem 73. *Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact.*

Theorem 72. *Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. If f is continuous at a point $c \in D$ and g is continuous at $f(c)$, then the composition $g \circ f : D \to \mathbb{R}$ is continuous at c.*

Lemma 4. *Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) < 0 < f(b)$. Then there exists a point c in (a, b) such that $f(x) = 0$.*

Corollary 5. *Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Then f assumes minimum and maximum values on D. That is, there exist points x_1 and x_2 in D such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in D$.*

Theorem 75. *Let I be a compact interval and suppose that $f : I \to \mathbb{R}$ is a continuous function. Then the set $f(I)$ is a compact interval.*

Theorem 74. *(Intermediate Value Theorem) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then f has the intermediate value property on $[a, b]$. That is, if k is any value between $f(a)$ and $f(b)$ [i.e. $f(a) < k < f(b)$ or $f(b) < k < f(a)$], then there exists $c \in [a, b]$ such that $f(c) = k$.*

Theorem 77. *Let $f : D \to \mathbb{R}$ be uniformly continuous on D and suppose that (x_n) is a Cauchy sequence in D. Then $(f(x_n))$ is a Cauchy sequence.*

Theorem 76. *Suppose that $f : D \to \mathbb{R}$ is continuous on a compact set D. Then f is uniformly continuous on D.*

Theorem 79. *Let I be an interval containing the point c and suppose that $f : I \to \mathbb{R}$. Then f is differentiable at c iff, for every sequence (x_n) in $I \setminus \{c\}$ that converges to c, the sequence*

$$\left(\frac{f(x_n) - f(c)}{x_n - c} \right)$$

converges. Furthermore, if f is differentiable at c, then the sequence of quotients above will converge to $f'(c)$.

Theorem 78. *A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b) iff it can be extended to a function $\tilde{f}$ that is continuous on $[a, b]$.*

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Theorem 80

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Theorem 81 (part 1)

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Theorem 81 (part 2)

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Theorem 82
Chain Rule

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Theorem 83

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Theorem 84
Rolle's Theorem

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Theorem 85
Mean Value Theorem

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Theorem 86

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COROLLARY

Corollary 6

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Theorem 87

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Theorem 81. *Suppose that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then*

(a) If $k \in \mathbb{R}$, then the function kf is differentiable at c and $(kf)'(c) = k \cdot f'(c)$.

(b) The function $f+g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$.

Theorem 80. *If $f : I \to \mathbb{R}$ is differentiable at a point $c \in I$, then f is continuous at c.*

Theorem 82. *(Chain Rule) Let I and J be intervals in $\mathbb{R}$, let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where $f(I) \subseteq J$, and let $c \in I$. If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

Theorem 81. *Suppose that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then*

(c) (Product Rule) The function fg is differentiable at c and $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.

(d) (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}.$$

Theorem 84. *(Rolle's Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and such that $f(a) = f(b) = 0$. Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem 83. *If f is differentiable on an open interval (a, b) and if f assumes its maximum or minimum at a point $c \in (a, b)$, then $f'(c) = 0$.*

Theorem 86. *Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*

Theorem 85. *(Mean Value Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Theorem 87. *Let f be differentiable on an interval I. Then*

(a) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on i, and

(b) if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.

Corollary 6. *Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there exists a constant C such that $f = g + C$ on $[a, b]$.*

THEOREM *Theorem 88* *Intermediate Value Theorem for Derivatives* REAL ANALYSIS I	THEOREM *Theorem 89* *Inverse Function Theorem* REAL ANALYSIS I
THEOREM *Theorem 90* *Cauchy Mean Value Theorem* REAL ANALYSIS I	THEOREM *Theorem 91* *L'Hospital's Rule* REAL ANALYSIS I
THEOREM *Theorem 92* *L'Hospital's Rule* REAL ANALYSIS I	THEOREM *Theorem 93* *Taylor's Theorem* REAL ANALYSIS I
THEOREM *Theorem 94* REAL ANALYSIS I	THEOREM *Theorem 95* REAL ANALYSIS I
THEOREM *Theorem 96* REAL ANALYSIS I	THEOREM *Theorem 97* REAL ANALYSIS I

Theorem 89. *(Inverse Function Theorem) Suppose that f is differentiable on an interval I and $f'(x) \neq 0$ for all $x \in I$. Then f is injective, f^{-1} is differentiable on $f(I)$, and $(f^{-1})'(y) = \frac{1}{f'(x)}$, where $y = f(x)$.*

Theorem 88. *(Intermediate Value Theorem for Derivatives) Let f be differentiable on $[a, b]$ and suppose that k is a number between $f'(a)$ and $f'(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = k$.*

Theorem 91. *(L'Hospital's Rule) Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $c \in [a, b]$ and $f(c) = g(c) = 0$. Suppose also that $g'(x) \neq 0$ for $x \in U$, where U is the intersection of (a, b) and some deleted neighborhood of c. If $\displaystyle\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$, with $L \in \mathbb{R}$, then $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)} = L$.*

Theorem 90. *(Cauchy Mean Value Theorem) Let f and g be functions that are continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

Theorem 93. *(Taylor's Theorem) Let f and its first n derivatives be continuous on $[a, b]$ and differentiable on (a, b), and let $x_0 \in [a, b]$. Then for each $x \in [a, b]$ with $x \neq x_0$ there exists a point c between x and x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

$$+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Theorem 92. *(L'Hospital's Rule) Let f and g be differentiable on (b, ∞). Suppose that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, and that $g'(x) \neq 0$ for $x \in (b, \infty)$. If $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$, where $L \in \mathbb{R}$, then $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.*

Theorem 95. *Let f be a bounded function on $[a, b]$. Then $L(f) \leq U(f)$.*

Theorem 94. *Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$ and Q is a refinement of P, then $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.*

Theorem 97. *Let f be a monotonic function on $[a, b]$. Then f is integrable.*

Theorem 96. *Let f be a bounded function on $[a, b]$. Then f is integrable iff for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.*

Theorem *Theorem 98* Real Analysis I	Theorem *Theorem 99* Real Analysis I
Theorem *Theorem 100* Real Analysis I	Theorem *Theorem 101* Real Analysis I
Corollary *Corollary 7* Real Analysis I	Theorem *Theorem 102* *The Fundamental Theorem of Calculus I* Real Analysis I
Theorem *Theorem 103* *The Fundamental Theorem of Calculus II* Real Analysis I	Theorem *Theorem 104* Real Analysis I
Theorem *Theorem 105* Real Analysis I	Theorem *Theorem 106* *Cauchy Criterion for Series* Real Analysis I

Theorem 99. *Let f and g be integrable functions on $[a, b]$ and let $k \in \mathbb{R}$. Then*

(a) kf is integrable and $\int_a^b kf = k \int f$, and

(b) $f + g$ is integrable and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Theorem 98. *Let f be a continuous function on $[a, b]$. Then f is integrable on $[a, b]$.*

Theorem 101. *Suppose that f is integrable on $[a, b]$ and g is continuous on $[c, d]$, where $f([a, b]) \subseteq [c, d]$. Then $g \circ f$ is integrable on $[a, b]$.*

Theorem 100. *Suppose that f is integrable on both $[a, c]$ and $[c, b]$. Then f is integrable on $[a, b]$. Furthermore, $\int_a^b f = \int_a^c f + \int_c^b f$.*

Theorem 102. *(The Fundamental Theorem of Calculus I) Let f be integrable on $[a, b]$. For each $x \in [a, b]$ let $F(x) = \displaystyle\int_a^x f(t)\,dt$. Then F is uniformly continuous on $[a, b]$. Furthermore, if f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.*

Corollary 7. *Let f be integrable on $[a, b]$. The $|f|$ is integrable on $[a, b]$ and $\left| \displaystyle\int_a^b f \right| \leq \displaystyle\int_a^b |f|$.*

Theorem 104. *Suppose that $\sum a_n = s$ and $\sum b_n = t$. Then $\sum (a_n + b_n) = s + t$ and $\sum (ka_n) = ks$, for every $k \in \mathbb{R}$.*

Theorem 103. *(The Fundamental Theorem of Calculus II) If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$, then $\displaystyle\int_a^b f' = f(b) - f(a)$.*

Theorem 106. *(Cauchy Criterion for Series) The infinite series $\sum a_n$ converges iff for each $\epsilon > 0$ there exists a number N such that if $n \geq m > N$, then $|a_m + a_{m+1} + \cdots + a_n| < \epsilon$.*

Theorem 105. *If $\sum a_n$ is a convergent series, then $\lim a_n = 0$.*

THEOREM *Theorem 107* *Comparison Test* REAL ANALYSIS I	THEOREM *Theorem 108* REAL ANALYSIS I
THEOREM *Theorem 109* *Ratio Test* REAL ANALYSIS I	THEOREM *Theorem 110* *Root Test* REAL ANALYSIS I
THEOREM *Theorem 111* *Integral Test* REAL ANALYSIS I	THEOREM *Theorem 112* *Alternating Series Test* REAL ANALYSIS I
THEOREM *Theorem 113* REAL ANALYSIS I	THEOREM *Theorem 114* *Ratio Criterion* REAL ANALYSIS I
THEOREM *Theorem 115* REAL ANALYSIS I	THEOREM *Theorem 116* *Weierstrass M-test* REAL ANALYSIS I

Theorem 108. *If a series converges absolutely, then it converges.*

Theorem 107. *(Comparison Test) Let* $\sum a_n$ *and* $\sum b_n$ *be infinite series of nonnegative terms. That is,* $a_n \geq 0$ *and* $b_n \geq 0$ *for all* n*. Then*

1. *If* $\sum a_n$ *converges and* $0 \leq b_n \leq a_n$ *for all* n*, then* $\sum b_n$ *converges.*
2. *If* $\sum a_n = +\infty$ *and* $0 \leq a_n \leq b_n$ *for all* n*, then* $\sum b_n = +\infty$.

Theorem 110. *(Root Test) Given a series* $\sum a_n$*, let* $\alpha = \limsup |a_n|^{\frac{1}{n}}$.

1. *If* $\alpha < 1$*, then the series converges absolutely.*
2. *If* $\alpha > 1$*, then the series diverges.*
3. *Otherwise,* $\alpha = 1$ *and the test gives no information about convergence or divergence.*

Theorem 109. *(Ratio Test) Let* $\sum a_n$ *be a series of nonzero terms.*

1. *If* $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$*, then the series converges absolutely.*
2. *If* $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$*, the the series diverges.*
3. *Otherwise,* $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$ *and the test gives no information about convergence or divergence.*

Theorem 112. *(Alternating Series Test) If* (a_n) *is a decreasing sequence of positive numbers and* $\lim a_n = 0$*, then the series* $\sum (-1)^{n+1} a_n$ *converges.*

Theorem 111. *(Integral Test) Let* f *be a continuous function defined on* $[0, \infty)$*, and suppose that* f *is positive and decreasing. That is, if* $x_1 < x_2$*, then* $f(x_1) \geq f(x_2) > 0$*. Then the series* $\sum (f(n))$ *converges iff* $\lim_{n \to \infty} \left(\int_1^n f(x)\, dx \right)$ *exists as a real number.*

Theorem 114. *(Ratio Criterion) The radius of convergence* R *of a power series* $\sum a_n x^n$ *is equal to* $\lim \left|\frac{a_n}{a_{n+1}}\right|$*, provided that this limit exists.*

Theorem 113. *Let* $\sum a_n x^n$ *be a power series and let* $\alpha = \limsup |a_n|^{\frac{1}{n}}$*. Define* R *by*

$$R = \begin{cases} \frac{1}{\alpha} & \text{if } 0 < \alpha < +\infty \\ +\infty & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha = +\infty \end{cases}.$$

Then the series converges absolutely whenever $|x| < R$ *and diverges whenever* $|x| > R$*. (When* $R = +\infty$ *we take this to mean that the series converges absolutely for all real* x*. When* $R = 0$ *then the series converges only at* $x = 0$*.)*

Theorem 116. *(Weierstrass M-test) Suppose that* (f_n) *is a sequence of functions defined on* S *and* (M_n) *is a sequence of nonnegative numbers such that* $|f_n(x)| \leq M_n$ *for all* $x \in S$ *and all* $n \in \mathbb{N}$*. If* $\sum M_n$ *converges, then* $\sum f_n$ *converges uniformly on* S*.*

Theorem 115. *Let* (f_n) *be a sequence of functiond defined on a subset* S *of* $\mathbb{R}$*. There exists a function* f *such that* (f_n) *converges to* f *uniformly on* S *iff the following condition (called the Cauchy criterion) is satisfied:*
For every $\epsilon > 0$ *there exists a number* N *such that* $|f_n(x) - f_m(x)| < \epsilon$ *for all* $x \in S$ *and all* $m, n > N$*.*

Theorem *Theorem 117* Real Analysis I	Corollary *Corollary 8* Real Analysis I
Theorem *Theorem 118* Real Analysis I	Corollary *Corollary 9* Real Analysis I
Theorem *Theorem 119* Real Analysis I	Corollary *Corollary 10* Real Analysis I
Theorem *Theorem 120* Real Analysis I	Theorem *Theorem 121* Real Analysis I
Theorem *Theorem 122* Real Analysis I	Corollary *Corollary 11* Real Analysis I

Corollary 8. *Let* $\sum_{n=0}^{\infty} f_n$ *be a series of functions defined on a set* S*. Suppose that each* f_n *is continuous on* S *and that the series converges uniformly to a function* f *on* S*. Then* $f = \sum_{n=0}^{\infty} f_n$ *si continuous on* S*.*

Theorem 117. *Let* (f_n) *be a sequence of continuous functions defined on a set* S *and suppose that* (f_n) *converges uniformly on* S *to a function* $f : S \to \mathbb{R}$*. Then* f *is continuous on* S*.*

Corollary 9. *Let* $\sum_{n=0}^{\infty} f_n$ *be a series of functions defined on an interval* $[a, b]$*. Suppose that each* f_n *is continuous on* $[a, b]$ *and that the series converges uniformly to a function* f *on* $[a, b]$*. Then* $\int_a^b f(x)\, dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)\, dx$*.*

Theorem 118. *Let* (f_n) *be a sequence of continuous functions defined on an interval* $[a, b]$ *and suppose that* (f_n) *converges uniformly on* $[a, b]$ *to a function* f*. Then* $\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx$*.*

Corollary 10. *Let* $\sum_{n=0}^{\infty} f_n$ *be a series of functions that converges to a function* f *on an interval* $[a, b]$*. Suppose that for each* n*,* f_n' *exists and is continuous on* $[a, b]$ *and that the series of derivatives* $\sum_{n=0}^{\infty} f_n'$ *is uniformly convergent on* $[a, b]$*. Then* $f'(x) = \sum_{n=0}^{\infty} f_n'(x)$ *for all* $x \in [a, b]$*.*

Theorem 119. *Suppose that* (f_n) *converges to* f *on an interval* $[a, b]$*. Suppose also that each* f_n' *exists and is continuous on* $[a, b]$*, and that the sequence* (f_n') *converges uniformly on* $[a, b]$*. Then* $\lim_{n\to\infty} f_n'(x) = f'(x)$ *for each* $x \in [a, b]$*.*

Theorem 121. *Let* $\sum a_n x^n$ *be a power series with radius of convergence* R*, where* $0 < R \le +\infty$*. If* $0 < K < R$*, then teh power series converges uniformly on* $[-K, K]$*.*

Theorem 120. *There exists a continuous function defined on* $\mathbb{R}$ *that is nowhere differentiable.*

Corollary 11. *Suppose that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *for* $x \in (-R, R)$*, where* $R > 0$*. Then for each* $k \in \mathbb{N}$*, the* k*th derivative* $f^{(k)}$ *of* f *exists on* $(-R, R)$ *and*

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \\ &= k!a_k + (k+1)!a_{k+1} + \frac{(k+2)!}{2!} a_{k+2} x^2 + \cdots . \end{aligned}$$

Furthermore, $f^{(k)}(0) = k!a_k$*.*

Theorem 122. *Suppose that a pwer series converges to a function* f *on* $(-R, R)$*, where* $R > 0$*. Then the series can be differentiated term by term, and the differentiated series converges on* $(-R, R)$ *to* f'*. That is, if* $f(x) = \sum_{n=0}^{\infty} a_n x^n$*, then* $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$*, and both series have the same radius of convergence.*

Corollary

Corollary 12

Real Analysis I

Theorem

Theorem 123

Real Analysis I

Corollary

Corollary 13

Real Analysis I

Theorem 123. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series with a finite positive radius of convergence* R. *If the series converges at* $x = R$, *then it converges uniformly on teh interval* $[0, R]$. *Similarly, if the series converges at* $x = -R$, *then it converges uniformly on* $[-R, 0]$.

Corollary 12. *If* $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ *for all* x *in some interval* $(-R, R)$, *where* $R > 0$, *then* $a_n = b_n$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Corollary 13. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *have a finite positive radius of convergence* R. *If the series converges at* $x = R$, *then* f *is continuous at* $x = R$. *If the series converges at* $x = -R$, *then* f *is continuous at* $x = -R$.