Real Analysis I

Definition

statement

Real Analysis I

Definition

sentential connectives

Real Analysis I

Definition

negation

Real Analysis I

Definition

conjunction

Real Analysis I

Definition

disjunction

Real Analysis I

Definition

implication or conditional

Real Analysis I

Definition

antecedant & consequent
hypothesis & conclusion

Real Analysis I

Definition

equivalence

Real Analysis I

Definition

negation of a conjunction

Real Analysis I

A sentence that can unambiguously be classified as true or false.

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Let p stand for a statement, then $\sim p$ (read *not* p) represents the logical opposite or **negation** of p.

not, and, or, if . . . then, if and only if

If p and q are statements, then the statement p *or* q (called the **disjunction** of p and q and denoted $\mathbf{p} \vee \mathbf{q}$) is true unless both p and q are false.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

If p and q are statements, then the statement p *and* q (called the **conjunction** of p and q and denoted $\mathbf{p} \wedge \mathbf{q}$) is true only when both p and q are true, and false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

If p, then q.

In the above, the statement p is called the **antecedant** or **hypothesis**, and the statement q is called the **consequent** or **conclusion**.

A statement of the form

if p *then* q

is called an **implication** or **conditional**.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$$\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$$

A statement of the form "p if and only if q" is the conjunction of two implications and is called an **equivalence**.

Definition

negation of a disjunction

Real Analysis I

Definition

negation of an implication

Real Analysis I

Definition

tautology

Real Analysis I

Definition

universal quantifier

Real Analysis I

Definition

existential quantifier

Real Analysis I

Definition

contrapositive

Real Analysis I

Definition

converse

Real Analysis I

Definition

inverse

Real Analysis I

Definition

contradiction

Real Analysis I

Definition

subset

Real Analysis I

$$\sim (p \Rightarrow q) \Leftrightarrow p \wedge (\sim q)$$

$$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$$

$$\forall\, x,\ p(x)$$

In the above statement, the **universal quantifier** denoted by $\forall$ is read "for all", "for each", or "for every".

A sentence whose truth table contains only T is called a **tautology**. The following sentences are examples of tautologies ($c \equiv$ contradiction):

$$\begin{array}{lcl} (p \Leftrightarrow q) & \Leftrightarrow & (p \Rightarrow q) \wedge (q \Rightarrow p) \\ (p \Rightarrow q) & \Leftrightarrow & (\sim q \Rightarrow \sim p) \\ (p \Rightarrow q) & \Leftrightarrow & [(p \wedge \sim q) \Rightarrow c] \end{array}$$

The implication $p \Rightarrow q$ is logically equivalent with its **contrapositive**:

$$\sim q \Rightarrow \sim p$$

$$\exists\, x \ni p(x)$$

In the above statement, the **existential quantifier** denoted by $\exists$ is read "there exists ...", "there is at least one ...". The symbol $\ni$ is just shorthand for "such that".

Given the implication $p \Rightarrow q$ then its **inverse** is

$$\sim p \Rightarrow \sim q$$

An implication is *not* logically equivalent to its inverse. The inverse is the contrapositive of the converse.

Given the implication $p \Rightarrow q$ then its **converse** is

$$q \Rightarrow p$$

But they are *not* logically equivalent.

Let A and B be sets. We say that A is a **subset** of B if every element of A is an element of B. In symbols, this is denoted

$$A \subseteq B \text{ or } B \supseteq A$$

A **contradiction** is a statement that is always false. Contradictions are symbolized by the letter c or by two arrows pointing directly at each other.

$$\Rightarrow\Leftarrow$$

Definition

proper subset

Real Analysis I

Definition

set equality

Real Analysis I

Definition

union, intersection, complement, disjoint

Real Analysis I

Definition

indexed family of sets

Real Analysis I

Definition

pairwise disjoint

Real Analysis I

Definition

ordered pair

Real Analysis I

Definition

Cartesian product

Real Analysis I

Definition

relation

Real Analysis I

Definition

equivalence relation

Real Analysis I

Definition

equivalence class

Real Analysis I

Let A and B be sets. We say that A is a **equal** to B if A is a subset of B and B is a subset of A.

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$$

Let A and B be sets. A is a **proper subset** of B if A is a subset of B and there exists an element in B that is not in A.

If for each element j in a nomempty set J there corresponds a set A_j, then

$$\mathscr{A} = \{A_j : j \in J\}$$

is called an **indexed family of sets** with J as the index set.

Let A and B be sets.

$$\begin{aligned} A \cup B &= \{x : x \in A \text{ or } x \in B\} \\ A \cap B &= \{x : x \in A \text{ and } x \in B\} \\ A \setminus B &= \{x : x \in A \text{ and } x \notin B\} \end{aligned}$$

If $A \cap B = \varnothing$ then A and B are said to be **disjoint**.

The **ordered pair** (a, b) is the set whose members are $\{a\}$ and $\{a, b\}$.

$$(a, b) = \{\{a\}, \{a, b\}\}$$

If $\mathscr{A}$ is a collection of sets, then $\mathscr{A}$ is called **pairwise disjoint** if

$$\forall\ A, B \in \mathscr{A},\ \text{where } A \neq B \text{ then } A \cap B = \varnothing$$

Let A and B be sets. A **relaton** between A and B is any subset R of $A \times B$.

$$a\mathsf{R}b \Leftrightarrow (a, b) \in \mathsf{R}$$

If A and B are sets,then the **Cartesian product** or **cross product** of A and B is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

The **equivalence class** of $x \in S$ with respect to an equivalence relation R is the set

$$E_x = \{y \in S : y\mathsf{R}x\}$$

A relation R on a set S is an **equivalence relation** if for all $x, y, z \in S$ it satisfies the following criteria:

1. $x\mathsf{R}x$ reflexivity
2. $x\mathsf{R}y \Rightarrow y\mathsf{R}x$ symmetry
3. $x\mathsf{R}y$ and $y\mathsf{R}z \Rightarrow x\mathsf{R}z$ transitivity

Theorem

partition

Real Analysis I

Definition

function between A and B

Real Analysis I

Definition

domain

Real Analysis I

Definition

range & codomain

Real Analysis I

Definition

surjective or onto

Real Analysis I

Definition

injective or 1–1

Real Analysis I

Definition

bijective

Real Analysis I

Definition

characteristic or indicator function

Real Analysis I

Definition

image and pre-image

Real Analysis I

Definition

composition of functions

Real Analysis I

Let A and B be sets. A **function between A and B** is a nonempty relation $f \subseteq A \times B$ such that

$$[(a,b) \in f \text{ and } (a,b') \in f] \Longrightarrow b = b'$$

A **partition** of a set S is a collection $\mathscr{P}$ of nonempty subsets of S such that

1. Each $x \in S$ belongs to some subset $A \in \mathscr{P}$.
2. For all $A, B \in \mathscr{P}$, if $A \neq B$, then $A \cap B = \varnothing$

A member of a set $\mathscr{P}$ is called a **piece** of the partition.

Let A and B be sets, and let $f \subseteq A \times B$ be a function between A and B. The **range** of f is the set of all second elements of members of f.

$$\text{rng } f = \{b \in B : \exists\, a \in A \ni (a,b) \in f\}$$

The set B is referred to as the **codomain** of f.

Let A and B be sets, and let $f \subseteq A \times B$ be a function between A and B. The **domain** of f is the set of all first elements of members of f.

$$\text{dom } f = \{a \in A : \exists\, b \in B \ni (a,b) \in f\}$$

The function $f : A \to B$ is **injective** or (1–1) if:

$$\forall\, a, a' \in A, \quad f(a) = f(a') \Longrightarrow a = a'$$

The function $f : A \to B$ is **surjective** or **onto** if $B = \text{rng } f$. Equivalently,

$$\forall\, b \in B, \quad \exists\, a \in A \ni b = f(a)$$

Let A be a nonempty set and let $S \subseteq A$, then the **characteristic function** $\chi_S : A \to \{0,1\}$ is defined by

$$\chi_S(a) = \begin{cases} 0 & a \notin S \\ 1 & a \in S \end{cases}$$

A function $f : A \to B$ is said to be **bijective** if f is both surjective and injective.

Suppose $f : A \to B$ and $g : B \to C$, then the **composition** of g with f denoted by $g \circ f : A \to C$ is given by

$$(g \circ f)(x) = g(f(x))$$

In terms of ordered pairs this means

$$g \circ f = \{(a,c) \in A \times C : \exists\, b \in B \ni (a,b) \in f \wedge (b,c) \in g\}$$

Suppose $f : A \to B$, and $C \subseteq A$, then the **image** of C under f is

$$f(C) = \{f(x) : x \in C\}$$

If $D \subseteq B$ then the **pre-image** of D in f is

$$f^{-1}(D) = \{x \in A : f(x) \in D\}$$

Definition

inverse function

Real Analysis I

Definition

identity function

Real Analysis I

Definition

equinumerous

Real Analysis I

Definition

finite & infinite sets

Real Analysis I

Definition

cardinal number & transfinite

Real Analysis I

Definition

denumerable

Real Analysis I

Definition

countable & uncountable

Real Analysis I

Definition

power set

Real Analysis I

Definition

continuum hypothesis

Real Analysis I

Definition

algebraic & transcendental

Real Analysis I

A function that maps a set A onto itself is called the **identity function** on A, and is denoted i_A.

If $f : A \to B$ is a bijection, then

$$\begin{aligned} f^{-1} \circ f &= i_A \\ f \circ f^{-1} &= i_B \end{aligned}$$

Let $f : A \to B$ be bijective. The **inverse function** of f is the function $f^{-1} : B \to A$ given by

$$f^{-1} = \{(y, x) \in B \times A : (x, y) \in f\}$$

A set S is said to be **finite** if $S = \varnothing$ or if there exists an $n \in \mathbb{N}$ and a bijection

$$f : \{1, 2, \ldots, n\} \to S.$$

If a set is not finite, it is said to be **infinite**.

Two sets S and T are **equinumerous**, denoted $S \sim T$, if there exists a bijection from S onto T.

A set S is said to be **denumerable** if there exists a bijection

$$f : \mathbb{N} \to S$$

Let $I_n = \{1, 2, \ldots, n\}$. The **cardinal number** of I_n is n. Let S be a set. If $S \sim I_n$ then S has n elements.

The cardinal number of $\varnothing$ is defined to be 0.

Finally, if a cardinal number is not finite, it is said to be **transfinite**.

Given any set S, the **power set** of S denoted by $\mathscr{P}(S)$ is the collection of all possible subsets of S.

If a set is finite or denumerable, then it is **countable**.

If a set is not countable, then it is **uncountable**.

A real number is said to be **algebraic** if it is a root of a polynomial with integer coefficients.

If a number is not algebraic, it is called **transcendental**.

Given that $|\mathbb{N}| = \aleph_0$ and $|\mathbb{R}| = c$, we know that $c > \aleph_0$, but is there any set with cardinality say λ such that $\aleph_0 < \lambda < c$?

The conjecture that there is no such set was first made by Cantor and is known as the **continuum hypothesis**.

Axiom

well–ordering property of $\mathbb{N}$

Real Analysis I

Definition

basis for induction, induction step, induction hypothesis

Real Analysis I

Definition

recursion relation or recurrence relation

Real Analysis I

Axiom

field axioms

Real Analysis I

Axiom

order axioms

Real Analysis I

Definition

absolute value

Real Analysis I

Theorem

triangle inequality

Real Analysis I

Definition

ordered field

Real Analysis I

Definition

irrational number

Real Analysis I

Definition

upper & lower bound

Real Analysis I

In the *Principle of Mathematical Induction*, part (1) which refers to $P(1)$ being true is known as the **basis for induction**.

Part (2) where one must show that $\forall\ k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$ is known as the **induction step**.

Finally, the assumption in part (2) that $P(k)$ is true is known as the **induction hypothesis**.

If S is a nonempty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $\forall\ k \in S\ m \leq k$.

A1 Closure under addition
A2 Addition is commutative
A3 Addition is associative
A4 Additive identity is 0
A5 Unique additive inverse of x is $-x$
M1 Closure under multiplication
M2 Multiplication is commutative
M3 Multiplication is associative
M4 Multiplicative identity is 1
M5 If $x \neq 0$, then the unique multiplicative inverse is $1/x$
DL $\forall\ x, y, z \in \mathbb{R}, x(y+z) = xy + xz$

A **recurrence relation** is an equation that defines a sequence recursively: each term of the sequence is defined as a function of the preceding terms.

The Fibonacci numbers are defined using the linear recurrence relation:

$$\begin{aligned} F_n &= F_{n-2} + F_{n-1} \\ F_1 &= 1 \\ F_2 &= 1 \end{aligned}$$

If $x \in \mathbb{R}$, then the **absolute value** of x, is denoted $|x|$ and defined to be

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

O1 $\forall\ x, y \in \mathbb{R}$ exactly one of the relations $x = y, x < y, x > y$ holds. (trichotomy)

O2 $\forall\ x, y, z \in \mathbb{R}$, $x < y$ and $y < z \Rightarrow x < z$. (transitivity)

O3 $\forall\ x, y, z \in \mathbb{R}$, $x < y \Rightarrow x + z < y + z$

O4 $\forall\ x, y, z \in \mathbb{R}$, $x < y$ and $z > 0 \Rightarrow xz < yz$.

If S is a field and satisfies (O1–O4) of the order axioms, then S is an **ordered field**.

Let $x, y \in \mathbb{R}$ then

$$|x + y| \leq |x| + |y|$$

alternatively,

$$|a - b| \leq |a - c| + |c - b|$$

Let S be a subset of $\mathbb{R}$. If there exists an $m \in \mathbb{R}$ such that $m \geq s \quad \forall\ s \in S$, then m is called an **upper bound** of S.

Similarly, if $m \leq s \quad \forall\ s \in S$, then m is called a **lower bound** of S.

Suppose $x \in \mathbb{R}$. If $x \neq \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, then x is **irrational**.

Definition

bounded

Real Analysis I

Definition

maximum & minimum

Real Analysis I

Definition

supremum

Real Analysis I

Definition

infimum

Real Analysis I

Axiom

Completeness Axiom

Real Analysis I

Definition

Archimedean ordered field

Real Analysis I

Definition

dense

Real Analysis I

Definition

extended real numbers

Real Analysis I

Definition

neighborhood & radius

Real Analysis I

Definition

deleted neighborhood

Real Analysis I

If m is an upper bound of S and also in S, then m is called the **maximum** of S.

Similarly, if m is a lower bound of S and also in S, then m is called the **minimum** of S.

A set S is said to be **bounded** if it is bounded above and bounded below.

Let S be a nonempty subset of $\mathbb{R}$. If S is bounded below, then the **greatest lower bound** is called the **infimum**, and is denoted $\inf S$.

$m = \inf S \Leftrightarrow$

(a) $m \leq s, \ \forall \ s \in S$ and

(b) if $m' > m$, then $\exists \ s' \in S \ni s' < m'$

Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then the **least upper bound** is called the **supremum**, and is denoted $\sup S$.

$m = \sup S \Leftrightarrow$

(a) $m \geq s, \ \forall \ s \in S$ and

(b) if $m' < m$, then $\exists \ s' \in S \ni s' > m'$

An ordered field F has the **Archimedean property** if

$$\forall \ x \in F \quad \exists \ n \in \mathbb{N} \ni x < n$$

Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

For convenience, we extend the set of real numbers with two symbols ∞ and $-\infty$, that is $\mathbb{R} \cup \{\infty, -\infty\}$.

Then for example if a set S is not bounded above, then we can write

$$\sup S = \infty$$

A set S is **dense** in a set T if

$$\forall \ t_1, t_2 \in T \quad \exists \ s \in S \ni t_1 < s < t_2$$

Let $x \in \mathbb{R}$ and $\varepsilon > 0$, then a **deleted neighborhood** of x is

$$N^*(x; \varepsilon) = \{y \in \mathbb{R} : 0 < |y - x| < \varepsilon\}$$

Let $x \in \mathbb{R}$ and $\varepsilon > 0$, then a **neighborhood** of x is

$$N(x; \varepsilon) = \{y \in \mathbb{R} : |y - x| < \varepsilon\}$$

The number ε is referred to as the **radius** of $N(x; \varepsilon)$.

Definition

interior point

Real Analysis I

Definition

closed and open sets

Real Analysis I

Definition

isolated point

Real Analysis I

Definition

open cover

Real Analysis I

Definition

compact set

Real Analysis I

Definition

boundary point

Real Analysis I

Definition

accumulation point

Real Analysis I

Definition

closure of a set

Real Analysis I

Definition

subcover

Real Analysis I

Definition

sequence

Real Analysis I

A point $x \in \mathbb{R}$ is a **boundary point** of S if

$$\forall\ \varepsilon > 0, \quad N(x;\varepsilon) \cap S \neq \varnothing \text{ and } N(x;\varepsilon) \cap (\mathbb{R} \setminus S) \neq \varnothing$$

In other words, every neighborhood of a boundary point must intersect the set S and the complement of S in $\mathbb{R}$.

The set of all boundary points of S is denoted bd S.

Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is an **interior point** of S if there exists a neigborhood $N(x;\varepsilon)$ such that $N \subseteq S$.

The set of all interior points of S is denoted int S.

Suppose $S \subseteq \mathbb{R}$, then a point $x \in \mathbb{R}$ is called an **accumulation point** of S if

$$\forall\ \varepsilon > 0, \quad N^*(x;\varepsilon) \cap S \neq \varnothing$$

In other words, every deleted neighborhood of x contains a point in S.

The set of all accumulation points of S is denoted S'.

Let $S \subseteq \mathbb{R}$. If bd $S \subseteq S$, then S is said to be **closed**.

If bd $S \subseteq \mathbb{R} \setminus S$, then S is said to be **open**.

Let $S \subseteq \mathbb{R}$. The **closure** of S is defined by

$$\text{cl } S = S \cup S'$$

In other words, the closure of a set is the set itself unioned with its set of accumulation points.

Let $S \subseteq \mathbb{R}$. If $x \in S$ and $x \notin S'$, then x is called an **isolated point** of S.

Suppose $\mathscr{G} \subseteq \mathscr{F}$ are both families of indexed sets that cover a set S, then since $\mathscr{G}$ is a subset of $\mathscr{F}$ it is called a **subcover** of S.

An **open cover** of a set S is a family or collection of sets whose union contains S.

$$S \subseteq \mathscr{F} = \{F_n : n \in \mathbb{N}\}$$

A **sequence** s is a function whose domain is $\mathbb{N}$. However, instead of denoting the value of s at n by $s(n)$, we denote it s_n. The ordered set of all values of s is denoted (s_n).

A set S is **compact** iff *every* open cover of S contains a finite subcover of S.

Note: This is a difficult definition to use because to show that a set is compact you must show that *every* open cover contains a finite subcover.

Definition

converge & diverge

Real Analysis I

Definition

bounded sequence

Real Analysis I

Definition

diverge to $+\infty$

Real Analysis I

Definition

diverge to $-\infty$

Real Analysis I

Definition

nondecreasing, nonincreasing & monotone

Real Analysis I

Definition

increasing & decreasing

Real Analysis I

Definition

Cauchy sequence

Real Analysis I

Definition

subsequence

Real Analysis I

Definition

subsequential limit

Real Analysis I

Definition

lim sup & lim inf

Real Analysis I

A sequence is said to be **bounded** if its range $\{s_n : n \in \mathbb{N}\}$ is bounded. Equivalently if,

$$\exists\, M \geq 0 \text{ such that } \forall\, n \in \mathbb{N},\ |s_n| \leq M$$

A sequence (s_n) is said to **converge** to $s \in \mathbb{R}$, denoted $(s_n) \to s$ if

$$\forall\, \varepsilon > 0,\ \exists\, N \text{ such that } \forall\, n \in \mathbb{N},$$

$$n > N \Rightarrow |s_n - s| < \varepsilon$$

If a sequence does not converge, it is said to **diverge**.

A sequence (s_n) is said to diverge to $-\infty$ if

$$\forall\, M \in \mathbb{R},\ \exists\, N \text{ such that}$$

$$n > N \Rightarrow s_n < M$$

A sequence (s_n) is said to diverge to $+\infty$ if

$$\forall\, M \in \mathbb{R},\ \exists\, N \text{ such that}$$

$$n > N \Rightarrow s_n > M$$

A sequence (s_n) is **increasing** if

$$s_n < s_{n+1} \quad \forall n \in \mathbb{N}$$

A sequence (s_n) is **decreasing** if

$$s_n > s_{n+1} \quad \forall n \in \mathbb{N}$$

A sequence (s_n) is **nondecreasing** if

$$s_n \leq s_{n+1} \quad \forall n \in \mathbb{N}$$

A sequence (s_n) is **nonincreasing** if

$$s_n \geq s_{n+1} \quad \forall n \in \mathbb{N}$$

A sequence is **monotone** if it is either nondecreasing or nonincreasing.

If (s_n) is any sequence and (n_k) is any strictly increasing sequence, then the sequence (s_{n_k}) is called a **subsequence** of (s_n).

A sequence (s_n) is said to be a **Cauchy sequence** if

$$\forall\, \varepsilon > 0, \quad \exists\, N \text{ such that}$$

$$m, n > N \ \Rightarrow |s_n - s_m| < \varepsilon$$

Suppose S is the set of all subsequential limits of a sequence (s_n). The **lim sup** (s_n), shorthand for the limit superior of (s_n) is defined to be

$$\text{lim sup } (s_n) = \text{sup } S$$

The **lim inf** (s_n), shorthand for the limit inferior of (s_n) is defined to be

$$\text{lim inf } (s_n) = \text{inf } S$$

A **subsequential limit** of a sequence (s_n) is the limit of some subsequence of (s_n).

Definition

oscillating sequence

Real Analysis I

Definition

limit of a function

Real Analysis I

Definition

sum, product, multiple, & quotient of functions

Real Analysis I

Definition

right–hand limit

Real Analysis I

Definition

left–hand limit

Real Analysis I

Definition

continuous function at a point

Real Analysis I

Definition

continuous on S
continuous

Real Analysis I

Definition

bounded function

Real Analysis I

Definition

uniform continuity

Real Analysis I

Definition

extension of a function

Real Analysis I

Suppose $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$, and suppose c is an accumulation point of D. Then the **limit of f at c is** L is denoted by

$$\lim_{x \to c} f(x) = L$$

and defined by

$$\forall\, \varepsilon > 0, \quad \exists\, \delta > 0 \text{ such that}$$

$$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

If lim inf $(s_n) <$ lim sup (s_n), then we say that the sequence (s_n) **oscillates**.

Let $f : (a, b) \to \mathbb{R}$, then the **right–hand limit** of f at a is denoted

$$\lim_{x \to a^+} f(x) = L$$

and defined by

$$\forall\, \varepsilon > 0, \quad \exists\, \delta > 0 \text{ such that}$$

$$a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$$

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, then we define:

1. **sum** $(f + g)(x) = f(x) + g(x)$
2. **product** $(fg)(x) = f(x)g(x)$
3. **multiple** $(kf)(x) = kf(x) \quad k \in \mathbb{R}$
4. **quotient** $\left(\dfrac{f}{g}\right) = \dfrac{f(x)}{g(x)}$ if $g(x) \neq 0 \quad \forall x \in D$

Let $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$, and suppose $c \in D$, then f is **continuous** at c if

$$\forall\, \varepsilon > 0, \quad \exists\, \delta > 0 \text{ such that}$$

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

Let $f : (a, b) \to \mathbb{R}$, then the **left–hand limit** of f at b is denoted

$$\lim_{x \to b^-} f(x) = L$$

and defined by

$$\forall\, \varepsilon > 0, \quad \exists\, \delta > 0 \text{ such that}$$

$$b - \delta < x < b \Rightarrow |f(x) - L| < \varepsilon$$

A function is said to be **bounded** if its range is bounded. Equivalently, $f : D \to \mathbb{R}$ is bounded if

$$\exists\, M \in \mathbb{R} \text{ such that } \forall\, x \in D,\ |f(x)| \leq M$$

Let $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$. If f is continuous at each point of a subset $S \subseteq D$, then f is said to be **continuous on** S.

If f is continuous on its entire domain D, then f is simply said to be **continuous**.

Suppose $f : (a, b) \to \mathbb{R}$, then the **extension of** f is denoted $\tilde{f} : [a, b] \to \mathbb{R}$ and defined by

$$\tilde{f}(x) = \begin{cases} u & x = a \\ f(x) & a < x < b \\ v & x = b \end{cases}$$

where $\lim\limits_{x \to a} f(x) = u$ and $\lim\limits_{x \to b} f(x) = v$.

A function $f : D \to \mathbb{R}$ is **uniformly continuous on** D if

$$\forall\, \varepsilon > 0, \quad \exists\, \delta > 0 \text{ such that}$$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Definition

differentiable at a point

Real Analysis I

Definition

strictly increasing function
strictly decreasing function

Real Analysis I

Definition

limit at ∞

Real Analysis I

Definition

tends to ∞

Real Analysis I

Definition

Taylor polynomials for f at x_0

Real Analysis I

Definition

Taylor series

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Definition

partition of an interval
refinement of a partition

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Definition

upper sum

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Definition

lower sum

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Definition

upper integral
lower integral

Real Analysis I

A function $f : D \to \mathbb{R}$ is said to be **strictly increasing** if

$$\forall\, x_1, x_2 \in D, \quad x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

A function $f : D \to \mathbb{R}$ is said to be **strictly decreasing** if

$$\forall\, x_1, x_2 \in D, \quad x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Suppose $f : I \to \mathbb{R}$ where I is an interval containing the point c. Then f is **differentiable at** c if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite. Whenever this limit exists and is finite, we denote the **derivative of** f **at** c by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Suppose $f : (a, \infty) \to \mathbb{R}$, then we say f **tends to** ∞ as $x \to \infty$ and denote it by

$$\lim_{x \to \infty} f(x) = \infty$$

iff

$$\forall\, M \in \mathbb{R}, \quad \exists\, N > a \text{ such that}$$
$$x > N \Rightarrow f(x) > M$$

Suppose $f : (a, \infty) \to \mathbb{R}$, then the **limit at infinity** of f denoted

$$\lim_{x \to \infty} f(x) = L$$

iff

$$\forall\, \varepsilon > 0, \quad \exists\, N > a \text{ such that}$$
$$x > N \Rightarrow |f(x) - L| < \varepsilon$$

If f has derivatives of all orders in a neighborhood of x_0, then the limit of the Taylor polynomials is an infinite series called the **Taylor series** of f at x_0.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots \end{aligned}$$

$$\begin{aligned} p_0(x) &= f(x_0) \\ p_1(x) &= f(x_0) + f'(x_0)(x - x_0) \\ p_2(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\vdots \\ p_n(x) &= f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \end{aligned}$$

Suppose f is a bounded function on $[a, b]$ and $P = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$.
For each $i \in \{1, \ldots, n\}$ let

$$M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

We define the **upper sum** of f with respect to P to be

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$.

A **partition** of an interval $[a, b]$ is a finite set of points $P = \{x_0, x_1, x_2, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \ldots < x_n = b$$

If P and P' are two partitions of $[a, b]$ where $P \subset P'$ then P' is called a **refinement** of P.

Suppose f is a bounded function on $[a, b]$. We define the **upper integral** of f on $[a, b]$ to be

$$U(f) = \inf\{U(f, P) : P \text{ any partition of } [a, b]\}.$$

Similarly, we define the **lower integral** of f on $[a, b]$ to be

$$L(f) = \sup\{L(f, P) : P \text{ any partition of } [a, b]\}.$$

Suppose f is a bounded function on $[a, b]$ and $P = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$.
For each $i \in \{1, \ldots, n\}$ let

$$m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

We define the **lower sum** of f with respect to P to be

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$.

Definition

Riemann integrable

Real Analysis I

Definition

monotone function

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Definition

proper integral

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Definition

improper integral

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Definition

integral convergence
integral divergence

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Definition

infinite series
partial sum

Real Analysis I

Definition

convergent series
sum

Real Analysis I

Definition

divergent series
diverge to $+\infty$

Real Analysis I

Definition

harmonic series

Real Analysis I

Definition

geometric series

Real Analysis I

A function is said to be **monotone** if it is either increasing or decreasing.

A function is increasing if $x < y \Rightarrow f(x) \leq f(y)$.
A function is decreasing if $x < y \Rightarrow f(x) \geq f(y)$.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If $L(f) = U(f)$, then we say f is **Riemann integrable** or just **integrable**. Furthermore,

$$\int_a^b f = \int_a^b f(x)dx = L(f) = U(f)$$

is called the **Riemann integral** or just the **integral** of f on $[a, b]$.

An **improper integral** is the limit of a definite integral, as an endpoint of the interval of integration approaches either a specified real number or ∞ or $-\infty$ or, in some cases, as both endpoints approach limits.

Let $f : (a, b] \to \mathbb{R}$ be integrable on $[c, b]\ \forall\ c \in (a, b]$. If $\lim_{c \to a^+} \int_c^b f$ exists then

$$\int_a^b f = \lim_{c \to a^+} \int_c^b f$$

When a function f is bounded and the interval over which it is integrated is bounded, then if the integral exists it is called a **proper integral**.

Let (a_k) be a sequence of real numbers, then we can create a new sequence of numbers (s_n) where each s_n in (s_n) corresponds to the sum of the first n terms of (a_k). This new sequence of sums is called an **infinite series** and is denoted by $\sum_{n=0}^{\infty} a_n$.

The n-th **partial sum** of the series, denoted by s_n is defined to be

$$s_n = \sum_{k=0}^{n} a_k$$

Suppose $f : (a, b] \to \mathbb{R}$ is integrable on $[c, b]\ \forall\ c \in (a, b]$, futhermore let $L = \lim_{c \to a^+} \int_c^b f$. If L is finite, then the improper integral $\int_a^b f$ is said to **converge** to L.

If $L = \infty$ or $L = -\infty$, then the improper integral is said to **diverge**.

If a series does not converge then it is **divergent**.

If the $\lim_{n \to \infty} s_n = +\infty$ then the series is said to **diverge to** $+\infty$.

If (s_n) converges to a real number say s, then we say that the series $\sum_{n=0}^{\infty} a_n = s$ is **convergent**.

Furthermore, we call s the **sum** of the series.

The **geometric series** is given by

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

The geometric series converges to $\dfrac{1}{1-x}$ for $|x| < 1$, and diverges otherwise.

The **harmonic series** is given by

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

The harmonic series diverges to $+\infty$.

Definition

converge absolutely
converge conditionally

Real Analysis I

Definition

power series

Real Analysis I

Definition

radius of convergence

Real Analysis I

Definition

interval of convergence

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Definition

converges pointwise

Real Analysis I

Definition

converges uniformly

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Definition

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Definition

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Definition

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Given a sequence (a_n) of real numbers, then the series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

is called a **power series**. The number a_n is called the n**th coefficient** of the series.

If $\sum |a_n|$ converges then the series $\sum a_n$ is said to **converge absolutely**.

If $\sum a_n$ converges, but $\sum |a_n|$ diverges, then the series $\sum a_n$ is said to **converge conditionally**.

The **interval of convergence** of a power series is the set of all $x \in \mathbb{R}$ such that $\sum_{n=0}^{\infty} a_n x^n$ converges.

By theorem we see that (for a power series centered at 0) this set will either be $\{0\}$, $\mathbb{R}$ or a bounded interval centered at 0.

The **radius of convergence** of a power series $\sum a_n x^n$ is an extended real number R such that (for a power series centered at x_0)

$$|x - x_0| < R \Rightarrow \sum a_n x^n \text{ converges.}$$

Note that R may be 0, $+\infty$ or any number between.

Let (f_n) be a sequence of functions defined on a subset S of $\mathbb{R}$. Then (f_n) **converges uniformly** on S to a function f defined on S if

$$\forall\, \varepsilon > 0, \quad \exists\, N \text{ such that } \forall\, x \in S$$

$$n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

Let (f_n) be a sequence of functions defined on a subset S of $\mathbb{R}$. Then (f_n) **converges pointwise** on S if for each $x \in S$ the sequence of numbers $(f_n(x))$ converges. If (f_n) converges pointwise on S, then we define $f : S \to \mathbb{R}$ by

$$f(x) = \lim_{n \to \infty} f_n(x)$$

for each $x \in S$, and we say that (f_n) converges to f pointwise on S.