1. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. random variables which have a probability density function, but suppose that we do not know what the density function is. We do have two candidate density functions, however, and we believe that one of them is the correct density. Call these functions $f(x)$ and $g(x)$ and assume that $f(x) > 0$ and $g(x) > 0$ for all $x$ for simplicity.

A standard tool to test which of the two density functions is correct is the likelihood ratio test. In this test, we consider the following product of ratios which we call $M_n$,

$$M_n = \prod_{i=1}^{n} \frac{f(X_i)}{g(X_i)},$$

and set a threshold $c > 0$. We accept the hypothesis that $f$ is the density if $M_n > c$ otherwise we reject this hypothesis (and believe that $g$ is the density).

For the rest of the problem, we will assume that the true density is $g$.

(a) Show that $M_n$ is a martingale in the filtration $\mathcal{F}_n$ generated by $X_1, \ldots, X_n$. What is $\mathbb{E}[M_n]$?

(b) There is a random variable $M_\infty$ so that $\lim_{n \to \infty} M_n = M_\infty$.

Why is this true?

(c) Show that

$$\lim_{n \to \infty} \frac{1}{n} \log M_n = \int_{-\infty}^{\infty} \log \left( \frac{f(x)}{g(x)} \right) g(x) dx$$

(so long as the right hand side makes sense).

(d) It is a fact that

$$\int_{-\infty}^{\infty} \log \left( \frac{f(x)}{g(x)} \right) g(x) dx < 0$$

unless $f = g$. What does this tell you about $M_\infty$?

(e) Explain the meaning of the last part of the problem in terms of our test of whether or not $f$ is the true density function.
2. Fix $p$ with $0 < p < 1$ and $p \neq 1/2$. Suppose that $X_i$ are i.i.d. random variables with
\[ P(X_i = 1) = p, \quad P(X_i = -1) = 1 - p \]
and let $S_n = \sum_{i=1}^{n} X_i$. We have seen this process before when studying countable state Markov chains. It is a biased random walk on $\mathbb{Z}$ started from zero.

(a) Show that
\[ M_n = \left( \frac{1 - p}{p} \right)^{S_n} \]
is a martingale in the filtration generated by $X_1, \ldots, X_n$.

(b) Fix integers $A$ and $B$ and let $A < 0 < B$. Set $T = \min\{ n : S_n = A \text{ or } S_n = B \}$ be the first time $S_n$ visits the set $\{A, B\}$. Show that $E[M_T] = 1$ and use this fact to compute the probability that the walk reaches $A$ before $B$. **Hint:** The event that the chain reaches $A$ before $B$ is the same as the event that $S_T = A$.

We did this computation previously using Markov chain techniques. Don’t copy that argument here. Use martingales.

(c) Use your answer to the previous part of the problem to compute the probability that the random walk ever reaches $-1$. **Hint:** This is like reaching $-1$ before “reaching” infinity.

(d) Using the previous parts of this problem, determine when the Markov chain is recurrent and when it is transient? Does this agree with what we saw previously?
3. Fix $p$ with $0 < p < 1$ and $p \neq 1/2$. Suppose that $X_i$ are i.i.d. random variables with
\[ P(X_i = 1) = p, \quad P(X_i = -1) = 1 - p \]
and let $S_n = \sum_{i=1}^{n} X_i$. This is the same setup as the previous problem.
Define
\[ M_n = S_n - (2p - 1)n \]
(a) Show that $M_n$ is a martingale in the filtration $\mathcal{F}_n$ generated by $X_1, \ldots, X_n$.
(b) Fix integers $A$ and $B$ and let $A < 0 < B$. Set $T = \min\{ n : S_n = A \text{ or } S_n = B \}$ be the first time $S_n$ visits the set $\{A, B\}$. Argue that $\mathbb{E}[M_T] = 0$.
(c) Use the previous part of the problem to compute $\mathbb{E}[T]$.
(d) Call the time it takes to reach $-1$ $T_{-1}$, i.e. $T_{-1} = \inf\{ n : X_n = -1 \}$. Compute $\mathbb{E}[T_{-1}]$. You may assume that if $p < 1/2$, then $\mathbb{E}[T_{-1}] < \infty$. 
4. Suppose that \( X_i \) are i.i.d. random variables with 
\[
\mathbb{P}(X_i = 1) = \frac{1}{2}, \quad \mathbb{P}(X_i = -1) = \frac{1}{2}
\]
and let \( S_n = \sum_{i=1}^{n} X_i \). This is the same setup as the previous two problems.

(a) Find constants \( a \) and \( b \) so that 
\[
M_n = S_n^4 - 6 \cdot n \cdot S_n^2 + a \cdot n^2 + b \cdot n
\]
is a martingale in the filtration \( \mathcal{F}_n \) generated by \( X_1, \ldots X_n \).

**Hint:** You need to find \( a \) and \( b \) so that \( \mathbb{E}[M_{n+1} - M_n | \mathcal{F}_n] = 0 \). Use the binomial theorem to expand \((S_n + X_{n+1})^4\) and \((S_n + X_{n+1})^2\).

(b) Fix an integer \( A \) and set \( T = \min\{n : S_n = A \text{ or } S_n = -A\} \) be the first time \( S_n \) visits the set \{\( A, -A \}\}. Compute \( \mathbb{E}[T^2] \).

**Hint:** We computed \( \mathbb{E}[T] \) in class.
5. Suppose that \( X_i \) are i.i.d. random variables with
\[
\mathbb{P}(X_i = 1) = 1/2, \quad \mathbb{P}(X_i = -1) = 1/2
\]
and let \( S_n = \sum_{i=1}^{n} X_i \). Call the moment generating function of \( X_i \) \( h(\lambda) \) and the log moment generating function of \( X_i \) \( g(\lambda) \):
\[
h(\lambda) = \mathbb{E}[e^{\lambda X_i}] = \frac{1}{2} e^{\lambda} + \frac{1}{2} e^{-\lambda}
\]
\[
g(\lambda) = \ln \mathbb{E}[e^{\lambda X_i}] = \ln \left( \frac{1}{2} e^{\lambda} + \frac{1}{2} e^{-\lambda} \right)
\]
(a) Show that for each \( \lambda \in \mathbb{R} \),
\[
M_n(\lambda) = e^{\lambda S_n - ng(\lambda)}
\]
is a martingale in the filtration \( \mathcal{F}_n \) generated by \( X_1, \ldots, X_n \).
(b) Let \( T = \min\{n : S_n = 1\} \) be the time when the random walk first reaches 1. Show that for \( \lambda \geq 0 \)
\[
1 = e^{\lambda} \mathbb{E}[h(\lambda)^{-T}].
\]
**Hint:** For all \( n \), \( S_{\min(n,T)} \leq 1 \) and \( g(\lambda) \geq 0 \) for all \( \lambda \). Use the optional stopping theorem with \( \min(T,n) \) and then dominated convergence or uniform integrability.
(c) Solve the equation
\[
\frac{1}{s} = \frac{1}{2} e^{-\lambda} + \frac{1}{2} e^{\lambda}
\]
to express \( e^{\lambda} \) as a function of \( s \) (hint: this is a quadratic polynomial). Use this computation and the previous parts of the problem to show that for \( s \in (0,1) \)
\[
\mathbb{E}[s^T] = \frac{1 - \sqrt{1 - s^2}}{s}.
\]