Please inform your instructor if you find any errors in the solutions.

1. Let $X$ be a uniform random variable on $[-1, 1]$. Let $Y = e^{-X}$. What is the probability density function of $Y$?

Solution: Since $X \in [-1, 1]$ we see that $e^{-X} \in [e^{-1}, e]$. We have

$$f_X(x) = \begin{cases} 
\frac{1}{2} & -1 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}.$$  

For $y \leq e^{-1}$,

$$P(Y \leq y) = P(e^{-X} \leq y) = 0.$$  

Similarly, if $y \geq e^{1}$, then

$$P(e^{-X} \leq y) = 1.$$  

If $y \in (e^{-1}, e)$, then

$$P(Y \leq y) = P(e^{-X} \leq y) = 1 - P(X \leq -\ln(y)).$$  

Differentiating, we see that for $y \in (e^{-1}, e)$,

$$f_Y(y) = \frac{1}{2y}.$$  

We see that

$$f_Y(y) = \begin{cases} 
\frac{1}{2y} & e^{-1} < y < e \\
0 & \text{otherwise}
\end{cases}.$$  

2. Let $X$ be an exponential random variable with parameter $\lambda > 0$. What is the probability density function of $Y = X^2$?

Solution: Recall that

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\
0 & x \leq 0
\end{cases}.$$  

Then if $y < 0$, we immediately see that

$$P(Y \leq y) = P(X^2 \leq y) = 0.$$  

$$P(Y \leq y) = P(e^{-X} \leq y) = 0.$$  

Similarly, if $y \geq e^{1}$, then

$$P(e^{-X} \leq y) = 1.$$  

If $y \in (e^{-1}, e)$, then

$$P(Y \leq y) = P(e^{-X} \leq y) = 1 - P(X \leq -\ln(y)).$$  

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2. Let $X$ be an exponential random variable with parameter $\lambda > 0$. What is the probability density function of $Y = X^2$?

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$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\
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\end{cases}.$$  

Then if $y < 0$, we immediately see that

$$P(Y \leq y) = P(X^2 \leq y) = 0.$$
If $y > 0$, then

$$P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}).$$

Differentiating leads to

$$f_Y(y) = \begin{cases} \frac{\lambda}{2\sqrt{y}} e^{-\lambda \sqrt{y}} & y > 0 \\ 0 & y \leq 0 \end{cases}. $$

3. Solve the following.

(a) (Log-normal distribution) Let $X$ be a standard normal random variable. Find the probability density function of $Y = e^X$.

(b) Let $X$ be a standard normal random variable nd let $Z$ be a random variable solution of $Z^3 + Z + 1 = X$. Find the probability density function of $Z$.

Solution:

(a) Let $g(x) = e^x$. Then $g$ is strictly increasing and so has an inverse as a function from $(-\infty, \infty)$ to $(0, \infty)$, which is given by $h(x) = g^{-1}(x) = \ln(x)$. Then

$$f_Y(y) = \begin{cases} \frac{1}{y\sqrt{2\pi}} e^{-\ln^2(y)/2} & y > 0 \\ 0 & y \leq 0 \end{cases}. $$

(b) Notice that for $h(x) = x^3 + x + 1$, $h$ is clearly continuous and we have $\lim_{x \to -\infty} h(x) = -\infty$ and $\lim_{x \to \infty} h(x) = \infty$. Moreover, since $h'(x) = 1 + 3x^2 > 0$, we see that $h$ is invertible. Call $g(x) = h^{-1}(x)$. Then we are looking for the probability density function of $Z = g(X)$. Since $h(x) = g^{-1}(x)$, we have for any $y \in (-\infty, \infty)$

$$f_Y(y) = f_X(h(y))|h'(y)| = \frac{1}{\sqrt{2\pi}} e^{-(y^3+y+1)^2/2} (1 + 3y^2).$$

4. Let $X$ be a continuous random variable with probability density function given by $f_X(x) = 1/x^2$ if $x \geq 1$ and 0 otherwise. A random variable $Y$ is given by

$$Y = \begin{cases} 2X & X \geq 2 \\ X^2 & X < 2 \end{cases}. $$

Find the probability density function of $Y$.

Solution: First we notice that the set of possible values of $Y$ is given by the set $(1, \infty)$. Define

$$g(x) = \begin{cases} 2x & x \geq 2 \\ x^2 & 0 < x < 2 \end{cases}. $$
Then $g$ is a strictly increasing and piecewise differentiable function from $(0, \infty)$ to $(0, \infty)$ with inverse given by

$$h(y) = \begin{cases} \frac{y}{2} & y \geq 4 \\ \sqrt{y} & 0 < y < 4 \end{cases}.$$  

$h$ is also a piecewise differentiable and strictly increasing function with

$$h'(y) = \begin{cases} \frac{1}{2} & y \geq 4 \\ \frac{1}{2\sqrt{y}} & 0 < y < 4 \end{cases}.$$  

We see then that

$$f_Y(y) = f_X(h(y))|h'(y)| = \begin{cases} \frac{2}{y^2} & y \geq 4 \\ \frac{1}{2y^{3/2}} & 1 \leq y < 4 \\ 0 & \text{otherwise} \end{cases}$$

5. Solve the following.

(a) Let $f$ be the probability density function of a continuous random variable $X$. Find the probability density function of $Y = X^2$.

(b) Let $X$ be a standard normal random variable. Show that $Y = X^2$ has a Gamma distribution and find the parameters.

**Solution:**

(a) Since $X$ is continuous and $X^2 \geq 0$, we see that for $y \leq 0$, we have $P(Y \leq y) = P(X^2 \leq y) = 0$. For $y > 0$,

$$P(Y \leq y) = P(X^2 \leq y) = P(|X| \leq \sqrt{y}) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

Differentiating, we see that

$$f_Y(y) = \begin{cases} \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} & y > 0 \\ 0 & y \leq 0 \end{cases}.$$  

(b) If $X$ is standard normal, then

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}}y^{1/2}e^{-y/2},$$

so $Y$ is a Gamma distribution with parameters $(1/2, 1/2)$.  