1. Suppose that we flip a fair coin. If the result of the flip is heads, we roll three dice. If the result is tails, we roll five dice. Let $H$ denote the number of heads we observe and $X$ denote the result of the number of dice which roll 2.

   (a) For $h = 0$ and $h = 1$, what is the conditional pmf of $X$ given $H = h$, $f_{X|H}(x|h)$?
   (b) Find the joint pmf of $X$ and $H$, $f_{X,H}(x,h)$.

**Solution:**

(a) 

\[
f_{X|H}(x|h) = \begin{cases} 
\frac{6}{x} \left( \frac{1}{6} \right)^x \left( \frac{5}{6} \right)^{6-x} & x = 0, 1, 2, 3, 4, 5, 6 \\
0 & \text{otherwise}
\end{cases}
\]

(b) 

\[
f_{X,H}(x,h) = f_{X|H}(x|0)f_H(0) + f_{X|H}(x|1)f_H(1) = \begin{cases} 
\frac{1}{2} \left( \frac{6}{x} \right) \left( \frac{1}{6} \right)^x \left( \frac{5}{6} \right)^{6-x} + \frac{3}{x} \left( \frac{1}{6} \right)^x \left( \frac{5}{6} \right)^{3-x} & x = 0, 1, 2, 3 \\
\frac{1}{2} \left( \frac{6}{x} \right) \left( \frac{1}{6} \right)^x \left( \frac{5}{6} \right)^{6-x} & x = 4, 5, 6 \\
0 & \text{otherwise}
\end{cases}
\]

2. Suppose that $X$ and $Y$ have joint mass function

\[
f_{X,Y}(x,y) = \begin{cases} 
\frac{1}{27} & (x,y) = (3,1) \\
\frac{2}{27} & (x,y) = (0,1) \\
\frac{6}{27} & (x,y) = (0,2), (1,2), (2,2), (1,3) \\
0 & \text{otherwise.}
\end{cases}
\]

   (a) Find the conditional mass function of $X$ given that $Y = 2$, $f_{X|Y}(x|2)$.
   (b) Find the conditional expectation $E[e^X|Y = 2]$.

**Solution:**

(a) 

\[
f_{X|Y}(x|2) = \begin{cases} 
\frac{1}{3} & x = 0, 1, 2 \\
0 & \text{otherwise}
\end{cases}
\]
(b) $E[e^X|Y = 2] = e^0(1/3) + e^1(1/3) + e^2(1/3)$.

3. Suppose that the joint mass function of $X$ and $Y$ is

$$f_{X,Y}(x, y) = \begin{cases} 
  a_1 & x = 0, y = 0 \\
  a_2 & x = 0, y = 1 \\
  a_3 & x = 0, y = 2 \\
  1/8 & x = 1, y = 0 \\
  a_4 & x = 1, y = 1 \\
  1/8 & x = 1, y = 2 \\
  0 & \text{otherwise} 
\end{cases}$$

Suppose that we know the following facts:

(a) Given that $X = 1$, $Y$ is uniform on $\{0, 1, 2\}$.
(b) $f_{X|Y}(0|0) = 2/3$.
(c) $E[Y|X = 0] = 4/5$.

Using this information, find $a_1, a_2, a_3,$ and $a_4$.

**Solution:** Since $Y$ is uniform on $\{0, 1, 2\}$ conditioned on $X = 1$, we must have $f_{Y|X}(0|1) = f_{Y|X}(1|1) = f_{Y|X}(2|1) = 1/3$. Since $f_{Y|X}(y|1) = f_{X,Y}(1,y)/f_X(1)$, we see that we must have

$$a_4 = 1/8$$

and

$$f_X(1) = f_{X,Y}(1,1)/f_{Y|X}(1|1) = 3/8$$

Since the possible values of $X$ are 0 and 1, it follows that $f_X(0) = 5/8$. Then

$$5/8 = f_{X,Y}(0,0) + f_{X,Y}(0,1) + f_{X,Y}(0,2) = a_1 + a_2 + a_3.$$  

We also have

$$2/3 = f_{X|Y}(0|0) = f_{X,Y}(0,0)/f_Y(0) = (a_1)/(a_1 + 1/8)$$

Rearranging this, we see that $a_1 = 1/4$. Finally, we see that

$$f_{Y|X}(0|1) = \frac{f_{X,Y}(0,1)}{f_X(0)} = \frac{a_2}{5/8}$$
$$f_{Y|X}(2|0) = \frac{f_{X,Y}(0,1)}{f_X(0)} = \frac{a_3}{5/8}$$

and therefore

$$4/5 = E[Y|X = 0] = 0f_{Y|X}(0|0) + 1f_{Y|X}(1|0) + 2f_{Y|X}(2|0) = 8a_2/5 + 16a_3/5.$$
We then have the system of equations
\[
\begin{align*}
3/8 &= a_2 + a_3 \\
4/5 &= 8a_2/5 + 16a_3/5
\end{align*}
\]
which has solution \(a_2 = 1/4\) and \(a_3 = 1/8\). To summarize
\[
a_1 = 1/4, a_2 = 1/4, a_3 = 1/8, a_4 = 1/8.
\]

4. Suppose that \(N\) is a random number which is drawn from a Binomial(10,1/4) distribution. Suppose that after observing that \(N = n\), we flip \(n\) fair coins. Let \(X\) be the number of heads that we observe from these fair coin flips.

(a) What is the conditional mass function of \(X\) given that \(N = n\), \(f_{X|N}(x|n)\)? (Be careful about the possible values of \(x\) and \(n\) here).

(b) For each possible value \(n\) of \(N\), compute \(E[X|N = n]\).

(c) Use the identity
\[
E[X] = \sum_{x,n} x f_{X,N}(x,n) = \sum_{x,n} x f_{X|N}(x|n) f_N(n) = \sum_n E[X|N = n] f_N(n)
\]
to compute \(E[X]\).

**Solution:**

(a) The possible values of \(N\) are \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\). For \(n = 1, 2, \ldots, 10\),
\[
f_{X|N}(x|n) = \begin{cases} \binom{n}{x} (1/2)^n & x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}
\]
and for \(n = 0\),
\[
f_{X|N}(x,0) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}
\]

(b) Since \(X\) is conditionally Binomial\((n,1/2)\) for \(n = 1, 2, \ldots, 10\),
\[
E[X|N = n] = n/2.
\]
and \(E[X|N = 0] = 0\) (= 0/2). The last observation is just to make the formula nicer.

(c)
\[
E[X] = \sum_{n=0}^{10} E[X|N = n] f_N(n) = \sum_{n=0}^{10} (n/2) f_N(n) = E[N/2] = (1/2) E[N] = (1/2)(10)(1/4) = 5/4.
\]
5. Suppose that $X_1, X_2,$ and $X_3$ are i.i.d. Bernoulli(1/4) random variables. Let $S_1 = X_1$, $S_2 = X_1 + X_2$ and $S_3 = X_1 + X_2 + X_3$.

(a) For all possible values of $k$, find the conditional probability mass functions of $S_3$ given $S_2 = k$ and $S_1 = k$: $f_{S_3|S_2}(x|k)$, $f_{S_3|S_1}(x|k)$.

(b) Compute $E[S_3|S_2]$ and $E[S_3|S_1]$.

Solution:

(a) The possible values of $S_2$ are 0, 1, 2. For $k = 0, 1, 2$,

$$f_{S_3|S_2}(x|k) = \begin{cases} 
3/4 & x = k \\
1/4 & x = k + 1 \\
0 & \text{otherwise}
\end{cases}$$

The possible values of $S_1$ are 0, 1. For $k = 0, 1$,

$$f_{S_3|S_1}(x|k) = \begin{cases} 
\binom{2}{0} (3/4)^2 & x = k \\
\binom{2}{1} (1/4)(3/4) & x = k + 1 \\
\binom{2}{2} (1/4)^2 & x = k + 2 \\
0 & \text{otherwise}
\end{cases}$$

(b) From the above computations, we see that for the possible values of $k$,

$$E[S_3|S_2 = k] = k + 1/4$$
$$E[S_3|S_1 = k] = k + 1/2$$

so

$$E[S_3|S_2] = S_2 + 1/4$$
$$E[S_3|S_1] = S_1 + 1/2.$$