Solve the following problems.

1. Suppose that $X$ is uniformly distributed on $[0, 2\pi]$ and let $Y = \cos(X)$ and $Z = \sin(X)$. Show that
   (a) $\text{Cov}(Y, Z) = 0$.
   (b) $Y$ and $Z$ are not independent.

2. Let $X$ and $Y$ be jointly continuous and uniformly distributed on the triangle with vertices at $(0,0)$, $(1,0)$, and $(0,1)$. Compute the correlation of $X$ and $Y$, $\rho(X,Y)$.

3. Suppose that $X$ and $Y$ are independent standard normal random variables. Fix $-1 < t < 1$ and let $Z = tX + \sqrt{1-t^2}Y$.
   (a) Compute the correlation of $X$ and $Z$, $\rho(X,Z)$.
   (b) Find the joint probability density function of $X$ and $Z$, $f_{X,Z}(x,z)$.

4. Suppose that I pick three numbers from the set $\{1, 2, 3\}$ uniformly at random, with replacement. Let $X$ denote the number of 2s I observe and $Y$ denote the number of different numbers I observe (so if my draw is 1,1,2, $Y = 2$).
   (a) Find the joint probability mass function of $X$ and $Y$, $f_{X,Y}(x,y)$.
   (b) Compute the correlation of $X$ and $Y$, $\rho(X,Y)$.

5. Suppose that $X$ is standard normal and $Z$ is independent of $X$ with $P(Z = 1) = P(Z = -1) = 1/2$. Let $Y = XZ$.
   (a) Show that $-X$ is standard normal.
   (b) Show that $Y$ is standard normal.
      Hints: We cannot use our previous change of variables formulas here because $Y$ and $Z$ are not jointly continuous ($Z$ is discrete). Instead, find the CDF of $Y$ by using the law of total probability with the events $A = \{Z = 1\}$ and $A^c = \{Z = -1\}$.
   (c) Show that $\text{Cov}(X,Y) = 0$.
   (d) Show that if $U$ and $V$ are independent standard normal random variables, then $P(U = V) = 0$.
   (e) Show that $X$ and $Y$ are not independent.
The point of this exercise is to provide a counterexample to a common misconception: even for normal random variables, being uncorrelated does not automatically imply independence.

6. Suppose that $X$, $Y$, and $Z$ are random variables with $E[X^2], E[Y^2], E[Z^2] < \infty$ which are all uncorrelated, i.e. $\text{Cov}(X, Y) = \text{Cov}(X, Z) = \text{Cov}(Y, Z) = 0$. Show that $\text{Var}(X + Y + Z) = \text{Var}(X) + \text{Var}(Y) + \text{Var}(Z)$.

7. Suppose that $\rho(X, Y) = t$ for some $|t| \leq 1$.
   
   (a) Show that if $a, b, c, d$ are non-random numbers with $a, c > 0$, then $\rho(aX + b, cY + d) = t$.
   
   (b) Suppose that $X$ and $Y$ have joint probability mass function
   
   \[
   f_{X,Y}(x, y) = \begin{cases} 
   1/4 & x = 2, y = 1 \\
   1/2 & x = 0, y = 1 \\
   1/4 & x = 0, y = 0 
   \end{cases}.
   \]
   
   Compute $\rho(X, Y)$ and $\rho(X^2, Y)$.

8. Suppose that $X$ is a standard normal random variable. Compute the following expectations and use these results along with Markov’s inequality to estimate $P(X \geq 6)$:

   (a) $E[X^2]$.
   
   (b) $E[e^X]$.
   
   (c) $E[e^{\frac{1}{4}X^2}]$.