Solve the following problems.

1. Fix \( p \in [0, 1] \) and consider a Markov chain \( X_n \) on the non-negative integers with the following dynamics: for all \( x \in \mathbb{Z}_+ \),

\[
\mathbb{P}(X_{n+1} = x + 1|X_n = x) = p, \quad \mathbb{P}(X_{n+1} = 0|X_n = x) = 1 - p.
\]


2. Suppose that \( X \) is a Markov chain on \( \mathbb{Z}_+ = \{0, 1, 2, 3, \ldots \} \) with transition probabilities

\[
\mathbb{P}(X_{n+1} = x + 2|X_n = x) = 1/2, \quad \mathbb{P}(X_{n+1} = x - 1|X_n = x) = 1/2,
\]

when \( x = 1, 2, 3, \ldots \), with \( \mathbb{P}(X_{n+1} = 2|X_n = 0) = 1/2 \) and \( \mathbb{P}(X_{n+1} = 0|X_n = 0) = 1/2 \).

(a) Draw the graph of this Markov chain.
(b) Determine if this Markov chain is recurrent or transient.

**Hint:** The general solution to

\[
f(x) = \frac{1}{2}f(x + 2) + \frac{1}{2}f(x - 1)
\]

for \( x = \{1, 2, 3, \ldots \} \) is

\[
f(x) = A + B \left( \frac{-1 + \sqrt{5}}{2} \right)^x + C \left( \frac{-1 - \sqrt{5}}{2} \right)^x
\]

3. Let \( p(x) \) be a probability mass function supported on \( \mathbb{Z}_+ \) (it is nonzero only at non-negative integers). Let \( X_n \) be a Markov chain with transition probabilities given for \( x \in \{1, 2, 3, 4, \ldots \} \) and \( y \in \{0, 1, 2, 3, \ldots \} \) by

\[
\mathbb{P}(X_{n+1} = x - 1|X_n = x) = 1, \quad \mathbb{P}(X_{n+1} = y|X_n = 0) = p(y).
\]

(a) Is this chain irreducible? If not, what are the communicating classes? Is it always aperiodic? If not, what are the possible periods of the different classes?
(b) Let \( T_0 = \min\{n \geq 1 : X_n = 0\} \) be the first return time to zero. Compute \( \mathbb{E}[T_0|X_0 = 0] \).
(c) When is this chain transient, null recurrent, or positive recurrent? Justify your answer.
4. Consider the following branching process $X_n$. Generation 0 has 1 individual, i.e. $X_0 = 1$. Then, each individual of each generation, independently of all other individuals, gives at most 2 descendants and immediately dies. Let $p \in [0, 1]$ be the probability an individual does not give any descendants and let $q \in [0, 1]$ be the probability the individual gives 2 descendants. Assume that $p + q \leq 1$.

(a) What is the condition on $p$ and $q$ for this process to have a chance at survival?
(b) Compute the probability of survival. (Hint: probability-generating function.)