1. Write code to simulate 100 samples, \( \{X_i\}_{i=1}^{100} \), of a random variable with probability mass function

\[
f(k) = \begin{cases} 
\frac{1}{2} & k = 1 \\
\frac{10000}{10000} & k = 10 \\
\frac{10000}{10000} & k = 100 \\
0 & \text{otherwise} 
\end{cases}
\]

(a) Use a computer or calculator to give a numerical approximation to the expectation of the exponential of one of these random variables: \( E[e^{X_1}] \).

(b) Compute the empirical approximation to this expectation: \( \sum_{i=1}^{100} e^{X_i} \).

2. (a) Suppose that \( f(x) \) is the probability density function

\[
f(x) = \frac{1}{2} e^{-|x|}
\]

Using the inverse transform method, write a program that samples 100 i.i.d. variables \( \{X_i\}_{i=1}^{\infty} \) with this probability density function and from your sample compute \( \frac{1}{100} \sum_{i=1}^{100} e^{\sin(X_i)} \).

(b) Using rejection sampling and your code in the previous part of the problem, sample 100 i.i.d. random variables \( \{Y_i\}_{i=1}^{\infty} \) which have density \( f(x) = Ce^{-x^4} \), where \( C \) is the unknown constant \( (\int_{-\infty}^{\infty} e^{-x^4} \, dx)^{-1} \). Compute \( \frac{1}{100} \sum_{i=1}^{100} Y_i^2 \).

(c) Suppose that \( p(x) \) is the probability mass function given by

\[
p(x) = \begin{cases} 
1/2 & x = 0 \\
1/8 & x = 1 \\
1/8 & x = 3 \\
0 & \text{otherwise} 
\end{cases}
\]

Write a program that samples 100 i.i.d. variables \( \{X_i\}_{i=1}^{100} \) with this probability mass function. From your sample, compute \( \frac{1}{100} \sum_{i=1}^{100} X_i \).

3. (a) Let \( \mathbb{P}(A_0 \cap A_1 \cap A_2 \cap A_3) > 0 \). Show that

\[
\mathbb{P}(A_0 \cap A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_0)\mathbb{P}(A_1|A_0)\mathbb{P}(A_2|A_0 \cap A_1)\mathbb{P}(A_3|A_2 \cap A_1 \cap A_0).
\]
(b) Consider a stochastic process \( \{X_0, X_1, \ldots \} \) on a state space \( S = \{a, b, c\} \). Let \( A_0 = \{X_0 = a\}, A_1 = \{X_1 = b\}, A_2 = \{X_2 = c\}, A_3 = \{X_3 = b\} \). What does the equation in the first part say about the probability \( P(X_0 = a, X_1 = b, X_2 = c, X_3 = b) \)?

(c) How does the previous expression simplify if \( X \) is a Markov chain?

4. Consider the following Markov chain on state space \( \{a, b, c\} \): the chain starts at \( a, b, \) or \( c \) with probability, respectively, 1/3, 1/6, and 1/2. Then, from \( a \) the chain either stays at \( a \) with probability 1/3, or goes to \( b \); from \( b \) the chain goes to \( a \) or \( c \) with probability 1/2 each; and from \( c \) the chain goes to \( a, b, \) or \( c \), with probabilities 1/4, 1/4, and 1/2, respectively.

(a) Write the transition matrix of the Markov chain.

(b) Using the formula you developed in the first problem, compute the probability \( P(X_0 = a, X_1 = b, X_2 = c, X_3 = b) \).

(c) Compute the probability \( P(X_1 = b) \) by considering all the possible scenarios leading to the event \( X_1 = b \).

(d) Compute \( P(X_0 = a|X_1 = b) \).

(e) Compute the probability \( P(X_5 = b) \). (Hint: compute powers of matrices.)

(f) Compute the invariant measure of the Markov chain exactly. (You can use a computer for computations. Do not just take a large power of the transition matrix!)

(g) Compute the limit of \( P(X_n = b) \) as \( n \to \infty \).