Please inform your instructor if you find any errors in the solutions.

1. You flip a fair coin 1000 times. Approximate the probability that the difference between the number of heads and number of tails is at most 10.

Solution: Call $X_i$ the Bernoulli random variable that is equal to 1 if the $i^{th}$ flip is heads. We are looking for

$$P \left( 495 \leq \sum_{i=1}^{1000} X_i \leq 505 \right)$$

The mean number of heads we would observe in 1000 trials is $np = (1000)(1/2) = 500$ and the standard deviation is $\sqrt{np(1-p)} = \sqrt{1000(1/2)(1/2)} = \sqrt{250} = 5\sqrt{10}$. Notice that $1/\sqrt{10} \approx .3162$. By the Central Limit Theorem, we then have

$$P \left( 495 \leq \sum_{i=1}^{1000} X_i \leq 505 \right) = P \left( -\frac{1}{\sqrt{10}} \leq \frac{\sum_{i=1}^{1000} X_i - 500}{\sqrt{250}} \leq \frac{1}{\sqrt{10}} \right) \approx \Phi(.32) - \Phi(-.32) \approx .251.$$ 

Above, we used the table in the back of the textbook to find $\Phi(.32) \approx 0.6255$ and $\Phi(-.32) = 1 - \Phi(.32) \approx 1 - 0.6255 \approx 0.3745$. Combining these, we have $0.6255 - 0.3745 = .251$.

2. We roll a die 72 times. Approximate the probability of getting exactly 3 sixes with both the normal and the Poisson approximation and compare the results with the exact probability 0.000949681.

Solution:

3. Suppose that $X$ is a random variable with moment generating function

$$M_X(t) = E[e^{tX}] = \begin{cases} \frac{1}{(1-5t)^3} & t < \frac{1}{5}, \\ \infty & t \geq \frac{1}{5}. \end{cases}$$

Compute $E[X]$.

Solution: We have

$$M_X(t) = (1 - 5t)^{-3},$$

$$M_X'(t) = 15(1 - 5t)^{-3}.$$ 

Then $E[X] = M_X'(0) = 15.$
4. Suppose that $X$ is a random variable with moment generating function

$$M_X(t) = \begin{cases} \frac{\lambda}{\lambda - t} & t < \lambda \\ \infty & t \geq \lambda \end{cases},$$

where $\lambda > 0$. For $n = 1, 2, \ldots$, find a formula for $E[X^n]$.

**Solution:** Using the geometric series formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

whenever $|x| < 1$. The Taylor-MacLaurin series for $M_X(t) = \frac{\lambda}{\lambda - t} = \frac{1}{1 - \frac{t}{\lambda}}$ is given by

$$M_X(t) = \sum_{n=0}^{\infty} \frac{M_X^{(n)}(0)}{n!} t^n = \sum_{n=0}^{\infty} \left( \frac{t}{\lambda} \right)^n = \sum_{n=0}^{\infty} \left( \frac{1}{\lambda} \right)^n t^n$$

which is valid for $t < \lambda$. Equating coefficients of $t^n$, we see that for $n = 0, 1, 2, \ldots$,

$$M_X^{(n)}(0) = \frac{n!}{\lambda^n}.$$

5. Let $X$ be a uniform random variable on $[-1, 1]$. Let $Y = e^{-X}$. What is the probability density function of $Y$?

**Solution:** Since $X \in [-1, 1]$ we see that $e^{-X} \in [e^{-1}, e]$. We have

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

For $y \leq e^{-1}$,

$$P(Y \leq y) = P(e^{-X} \leq y) = 0.$$ 

Similarly, if $y \geq e^1$, then

$$P(e^{-X} \leq y) = 1.$$ 

If $y \in (e^{-1}, e)$, then

$$P(Y \leq y) = P(e^{-X} \leq y) = 1 - P(X \leq -\ln(y)).$$

Differentiating, we see that for $y \in (e^{-1}, e)$,

$$f_Y(y) = \frac{1}{2y}.$$

We see that

$$f_Y(y) = \begin{cases} \frac{1}{2y} & e^{-1} < y < e \\ 0 & \text{otherwise} \end{cases}.$$
6. Let \( X \) be an exponential random variable with parameter \( \lambda > 0 \). What is the probability density function of \( Y = X^2 \)?

**Solution:** Recall that

\[
f_X(x) = \begin{cases} 
\lambda e^{-\lambda x} & x > 0 \\
0 & x \leq 0 
\end{cases}.
\]

Then if \( y < 0 \), we immediately see that

\[
P(Y \leq y) = P(X^2 \leq y) = 0.
\]

If \( y > 0 \), then

\[
P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}).
\]

Differentiating leads to

\[
f_Y(y) = \begin{cases} 
\frac{\lambda}{2\sqrt{y}} e^{-\lambda \sqrt{y}} & y > 0 \\
0 & y \leq 0 
\end{cases}.
\]