1. We roll a fair die three times. Let X be the number of times we roll a 6. What is the probability mass function of X?

Solution: If we think of rolling a 6 as a success then then $X$ has a Binomial($3, 1/6$) distribution.

$$f(0) = \left(\frac{5}{6}\right)^3$$

$$f(1) = \binom{3}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2$$

$$f(2) = \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)$$

$$f(3) = \left(\frac{1}{6}\right)^3 .$$

and $f(x) = 0$ for all other values of $x$.

2. An urn contains 5 balls numbered from 1 to 5. We draw 3 of them at random without replacement.

(a) Let $X$ be the largest number drawn. What is the probability mass function of $X$?

(b) Let $X$ be the smallest number drawn. What is the probability mass function of $X$?

Solution: There are $\binom{5}{3}$ possible outcomes in this experiment. We count the number of ways for each of these to happen in order to determine the probabilities.

(a) The possible values here are 3, 4 and 5, since we are drawing without replacement. In order for $k \in \{3, 4, 5\}$ to be the minimum, we must choose $k$ and then two numbers smaller than $k$. There are $\binom{k-1}{2}$ ways to do this, so

$$f(k) = P(X = k) = \frac{\binom{k-1}{2}}{\binom{5}{3}} \text{ if } k \in \{3, 4, 6\}$$

and $f(x) = 0$ otherwise.

(b) The possible values of $k$ are $\{1, 2, 3\}$. In order for $k \in \{1, 2, 3\}$ to be the minimum, we need to pick $k$ and two numbers larger than $k$. There are $\binom{5-k}{2}$ ways to do this, so

$$f(k) = P(X = k) = \frac{\binom{5-k}{2}}{\binom{3}{3}} \text{ if } k \in \{1, 2, 3\},$$

and $f(x) = 0$ for other values of $x$. 
3. Of the 100 people in a certain village, 50 always tell the truth, 30 always lie and 20 always refuse the answer. A sample size of 30 is taken without replacement.

(a) Find the probability that the sample will contain 10 people out of each category.
(b) Find the probability that there are exactly 12 liars.

Solution:

(a) 
\[
\binom{50}{10} \binom{20}{10} \binom{30}{10} \binom{100}{30}
\]

(b) 
\[
\binom{30}{12} \binom{70}{18} \binom{100}{30}
\]

4. We roll two fair dice.

(a) Let \( X \) be the product of the two outcomes. What is the probability mass function of \( X \)?
(b) Let \( X \) be the maximum of the two outcomes. What is the probability mass function of \( X \)?

Solution:

(a) By considering all products that can be formed by multiplying two numbers in \( \{1, 2, 3, 4, 5, 6\} \), we see that the possible values of \( X \) are 1,2,3,4,5,6,8,9, 10,12,15,16,18, 20,24,25, 30,36. Since all pairs are equally likely, we just need to count the number of pairs that multiply together to achieve each of these.

First, we notice that in order to obtain 1, 9, 16, 25, and 36, both dice have to be the same, so there is one way to obtain each of these.

To obtain a 4, we can either roll 2 twice or one die can be equal to 1 and one can be equal to 4, so there are three ways to do this.

To obtain a 6, we can either roll 2 and a 3 or a 3 and a 1, so there are four ways to do this. Similarly, to obtain a 12, we can either roll a 3 and a 4 or a 6 and a 2, so there are four ways to do this.

To obtain a 2, a 3, or a 5, we must roll a 1 and that number, so there are two ways to do each of these. Similarly, to obtain an 8, we must roll a 2 and a 4. To obtain a 10, we must roll a 5 and a 2. To obtain a 15, we must roll a 5 and a 3. To obtain an 18, we must roll a 6 and a 3. To roll a 20, we must roll a 5 and a 4. To obtain a 24, we must roll a 4 and a 6, and to obtain a 30, we must roll a 5 and a 6. For all of these cases, there are two ways to do this.
Combining these, we see that

\[
f(k) = \begin{cases} 
\frac{1}{36} & k = 1, 9, 16, 25, 36 \\
\frac{3}{36} & k = 4 \\
\frac{4}{36} & k = 6, 12 \\
\frac{2}{36} & k = 2, 3, 5, 8, 10, 15, 18, 20, 24, 30 \\
0 & \text{otherwise}
\end{cases}
\]

(b) The possible values here are 1, 2, 3, 4, 5, 6. Once again, all combinations are equally likely, so we only need to count how many combinations make each maximum possible. For \( k \in \{1, 2, 3, 4, 5, 6\} \), in order for \( k \) to be the maximum value, we need to roll \( k \) and a number less than or equal to \( k \). There is one way to roll \( k \) twice and there are \( 2(k - 1) \) ways to roll \( k \) and a number strictly less than \( k \). We see that the number of combinations which result in a maximum of \( k \) are \( 1 + 2(k - 1) \) and therefore for \( k \in \{1, 2, 3, 4, 5, 6\} \),

\[
f(k) = P(X = k) = \frac{1 + 2(k - 1)}{36}
\]

and for any \( x \) that is not in \( \{1, 2, 3, 4, 5, 6\} \), \( f(x) = 0 \).