1. A conference room contains \( m \) men and \( w \) women. These people seat at random in \( m + w \) seats arranged in a row. Find the probability that all the women will be adjacent.

**Solution:** There are \( (m + w)! \) possible ways to seat the \( m + w \) people in a row. To determine how to seat all the women together, we can pick how many men we place to the left of the row of women. Our options are any number in \( \{0, 1, 2, \ldots, m\} \), so there are \( m + 1 \) possible choices. We then have to consider all possible orderings of the men, of which there are \( m! \) choices, and all possible orderings of the women, of which there are \( w! \) choices. The answer is then

\[
\frac{(m + 1)m!w!}{(m + w)!}
\]
(c) We first pick which suit we are going to choose, of which there are 4 choices. Then we pick which card will be low (i.e. card with the smallest rank), of which there are 10 choices (the only possibilities are ace or a number in \{2, \ldots, 10\}). The probability is then
\[
\frac{40}{\binom{52}{5}}
\]

(d) We will first count all flushes and then subtract off the straight flushes, which we just counted. To obtain a flush, we first pick a suit, of which there are 4 choices. Then we pick which five ranks of that suit we will use, of which there are \(\binom{13}{5}\) choices. Subtracting off the number of straight flushes, we see that the probability is
\[
\frac{4 \cdot \binom{13}{5} - 40}{\binom{52}{5}}
\]

(e) We will follow the same outline as in part (d). First we count all straights and then we subtract off those which are straight flushes. We pick the low card as in part (c), of which there are 10 choices. Then we pick the \(4^5\) suits of these five cards (each card has one of four suits). Subtracting off the number of straights that are also flushes, we see that the probability is
\[
\frac{10 \cdot 4^5 - 40}{\binom{52}{5}}
\]

3. An experiment consists of drawing 10 cards from an ordinary 52-card deck.

(a) If the drawing is made with replacement, find the probability that no two cards have the same face value.

(b) If the drawing is made without replacement, find the probability that at least 9 cards will have the same suit.

**Solution:** We will assume throughout this problem that the order of the cards does not matter.

(a) If the drawing is made with replacement then there are \(52^{10}\) possible realizations if order matters. There are \(13!/3!\) possible choices of faces for the cards and then \(4^{10}\) ways to choose their suits. The probability is then
\[
\frac{13! \cdot 4^{10}}{3! \cdot 52^{10}}
\]

(b) Without replacement, there are \(\binom{52}{10}\) ways to select 10 cards from the deck. If we want at least nine of the cards to have the same suit we will break the problem into picking exactly nine and exactly ten cards of the same suit. In either case, we first pick the suit, of which there are 4 choices. To pick exactly nine cards, we then choose 9 ranks from among the 13 ranks and then pick a card from the \(52 - 13 = 39\) cards
remaining in the deck. There are $\binom{13}{9} \cdot 39$ ways to do this. Similarly, there are $\binom{13}{10}$ ways to choose ten cards of any given suit. The probability is then

$$\frac{4 \cdot \binom{13}{9} \cdot 39 + 4 \cdot \binom{13}{10}}{\binom{52}{10}}.$$ 

4. An urn contains 10 balls numbered 1 to 10. We draw five balls from the urn, *without* replacement. Find the probability that the second largest number drawn is 8.

**Solution:** Order does not matter for this problem, so there are $\binom{10}{5}$ possible choices of balls. To have 8 be the second highest choice, we need to pick one ball from among \{9, 10\} and three balls from \{1, 2, 3, 4, 5, 6, 7\}. There are 2 ways to do the former and $\binom{7}{3}$ ways to do the latter. The probability is then

$$\frac{2 \cdot \binom{7}{3} \cdot \binom{10}{5}}{\binom{10}{5}}.$$ 

5. (*The game of rencontre*). An urn contains $n$ tickets numbered 1, 2, ..., $n$. The tickets are shuffled thoroughly and then drawn one by one without replacement. If the ticket numbered $r$ appears in the $r$-th drawing, this is denoted as a match (French: *rencontre*). Show that the probability of at least one match is $1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1} \frac{1}{n!} \to 1 - e^{-1}$ as $n \to \infty$.

**Solution:** Let’s let $A_i$ be the event that we have a match at the $i$-th drawing. We are interested in computing the probability of at least one of the events $A_i$ occurring. This is $P\left(\bigcup_{i=1}^{n} A_i\right)$. By the inclusion-exclusion formula (see page 18 of the notes), we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leq j_1 < j_2 < \cdots < j_{i-1} < j_i} P(A_{j_1} \cap \cdots \cap A_{j_i}).$$

So now we just need to compute what $P(A_{j_1} \cap \cdots \cap A_{j_i})$ is, for all choices of subsets $j_1 < j_2 < \cdots < j_i$ taken from \{1, ..., $n$\} without replacement. Note that given any subset \{k_1, \ldots, k_i\} of size $i$ taken from \{1, ..., $n$\} without replacement and where order does not matter, we can always order them so that $k_1 < k_2 < \cdots < k_n$. For counting purposes, we see that there are the same number of subsets \{j_1, \ldots, j_i\} of \{1, 2, \ldots, $n$\} which satisfy $j_1 < j_2 < \cdots < j_i$ as there are subsets taken from \{1, 2, \ldots, $n$\} without replacement and where order does not matter. That is, there are $\binom{n}{i}$ terms in the second sum above.

The actual experiment here of drawing $n$ numbers and then recording those values is the same as drawing $n$ numbers from the set \{1, 2, \ldots, $n$\} without replacement, where order matters. There are $n!$ ways to do this. Now suppose that I fix the number $i$. The event $A_{j_1} \cap \cdots \cap A_{j_i}$ means that on the $j_{i}^{th}$ draw, I pick the card labelled $j_1$, on the $j_2^{th}$ draw, I pick the card labelled $j_2$, and so on up to $j_i$. For example, suppose that $n = 5$, $i = 3$, $j_1 = 1$, $j_2 = 3$, $j_3 = 5$, then $A_{j_1} \cap A_{j_2} \cap A_{j_3}$ is the event that the first card I draw is the card labelled ‘1’, the third card I draw is the card labelled ‘3’ and the fifth card
I draw is the card labelled ‘5’. Notice that I have not said anything about what happens to the other cards, which can be in any configuration (there can even be more matches). Since we are fixing $i$ of the values, there are $(n - i)!$ ways to arrange the remaining cards. Importantly this probability is the same for all choices of $j_1, \ldots, j_i$. From this, we see that for any $i$ and for any choice $j_1, \ldots, j_i$ with $j_1 < j_2 < \cdots < j_i$, we have

$$P(A_{j_1} \cap \cdots \cap A_{j_i}) = \frac{(n - i)!}{n!}.$$ 

Therefore,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leq j_1 < j_2 < \cdots < j_{i-1} < j_i} P(A_{j_1} \cap \cdots \cap A_{j_i})$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \frac{(n - i)!}{n!}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!(n-i)!} \frac{(n - i)!}{n!}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i!} = 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{n!}.$$